Decomposition and Enumeration in Partially Ordered Sets

by

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B.A. Harvard University, 1995

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1999

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Abstract

We study posets by finding chain decompositions. For several classes of posets, we obtain flag f-vector formulas from chain decompositions which are related to shellings. In each case, the generating function for the flag f-vector is a symmetric function, and we exploit the fact that different symmetric function bases may prove convenient for different classes of posets and that a particular basis may nicely reflect combinatorial, topological or representation-theoretic information about a poset. We use skew-Schur functions to study shuffle posets, elementary symmetric functions for noncrossing partition lattices for the classical reflection groups and power-sum symmetric functions for graded monoid posets. Our decompositions for shuffle posets and for noncrossing partition lattices of all types are into regions of the order complex which each contribute a skew-Schur function to the generating function for the flag f-vector, so this shows that the generating function is Schur-positive in each of these cases. The analysis of graded monoid posets also led us to generalize the partition lattice Π_n to posets of multiset partitions. We interpret coefficients in our flag f-vector formula for graded monoid posets as the Euler characteristic of cell complexes closely related to the order complex of multiset partition posets; we prove that these cell complexes are shellable and deduce Euler characteristic formulas in some special cases.

Our starting point was work of Simion and Stanley on posets possessing what are known as local symmetric group actions on maximal chains. We examine how the Frobenius characteristic of such an action may relate to the flag f-vector of the poset. In particular, we prove that the orbits of local symmetric group actions on lattices are always products of chains. We also provide new examples of local symmetric group actions. The chain decompositions we give are primarily based on the topology of order complexes, but we show how (in some cases) these specialize to decompositions into local symmetric group action orbits and also to symmetric boolean decompositions which in turn yield symmetric chain decompositions.

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Acknowledgments

I would like to thank Clara Chan for all sorts of good advice, including recommending studying combinatorics at MIT, and I want to reiterate my thanks to Persi Diaconis and Joe Harris for much-appreciated encouragement while I was in college. I wish to thank fellow (and recent) graduate students Phil Bradley, Ken Fan, Colin Ingalls, Robert Kleinberg and Dana Pascovici for many interesting discussions. I thank Sergey Fomin for teaching two of my favorite classes at MIT, Irena Peeva for helpful advice and Rodica Simion for her generous enthusiasm.

I want to thank Paul Edelman for asking about the topological implications of some of my work and then (together with Vic Reiner) spending an afternoon helping me get started in this direction. More generally, I wish to thank Vic Reiner for a great deal of good advice and well-timed encouragement thoughout my graduate career, and also for explaining his point of view on various things such as connections between subspace arrangements, Gröbner bases and monoid posets.

And of course I wish to thank my advisor, Richard Stanley, for many things: for helpful discussions, for pointers to all sorts of useful references, for taking time reading my thesis and making numerous helpful suggestions on presentation, for encouragement and for all his guidance. Finally, I thank my friends Dave Amundsen, Norm Peralta, Edith Wun and my parents for all their encouragement, and I am grateful to the Hertz Foundation for their generous support of my research through a graduate fellowship.

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Chapter 1

Introduction

This thesis studies poset decomposition as a means of giving a fairly unified description of poset structure. Our intention is to find ways to take advantage of the fact that many classes of posets which come up in practice are endowed with a great deal of structure. Therefore, we demonstrate how chain decompositions yield combinatorial, topological and representation-theoretic information in several classes of posets. Our chain decompositions for shuffle posets of multisets, k-shuffle posets and noncrossing partition lattices for the classical reflection groups lead to numerous applications, beginning with formulas for the flag f-vector. Graded monoid posets are in some sense a much harder class of examples, but we do still provide chain decompositions which yield flag f-vector formulas in terms of Gröbner bases and (generalized) Möbius functions of intersection lattices of subspace arrangements.

While chain decomposition is our primary focus, Chapters 5 and 6 address two other topics which are only indirectly related. Chapter 5 examines what are known as local symmetric group actions on the maximal chains in posets, characterizing possible orbits of local symmetric group actions on lattices. Chapter 5 goes on to give chain-labellings which induce local symmetric group actions and to indicate how some of the chain decompositions discussed in earlier chapters specialize to the orbits of these actions. Chapter 6 generalizes the lattice Π_n of partitions of $\{1,\ldots,n\}$ by dropping the assumption that the letters to be partitioned all need to be distinguishable. Our original motivation came from the fact that the coefficients in flag f-vector formulas for graded monoid posets may be viewed as Euler characteristics of cell complexes closely related to the order complexes of these multiset partition posets. Chapter 6 therefore gives Euler characteristic formulas in several cases. However, most of Chapter 6 is devoted to proving that these cell complexes are always shellable (despite the fact that the related order complexes are not necessarily even Cohen-Macaulay [Zi, p.218]). In the process, we extend the notion of lexicographic shellability in two directions. In the special case of the partition lattice, the cell complexes actually agree with the order complex and our shelling turns into a lexicographic shelling of the dual poset to the partition lattice.

Appendix A provides terminology and background on partially ordered sets, topological combinatorics, quasi-symmetric functions and symmetric functions in an effort to make this thesis fairly self-contained. Alternatively, we suggest [Bj3] as a very helpful reference on topological combinatorics, [Sa], [St6] or [Ma] for useful background on symmetric functions and [St3] and [St6] as good general references on combinatorics.

Chapter 2 defines several classes of posets for which Chapter 3 gives chain decompositions and consequent flag f-vector formulas. Chapter 4 provides further applications. Each decomposition comes from breaking a poset into overlapping pieces, partially ordering these pieces and then assigning each chain to the earliest piece containing it. The decompositions are topological in the sense that the pieces tend to come from cycles comprising a homology basis, and in general the partial order on these pieces is related to a shelling.

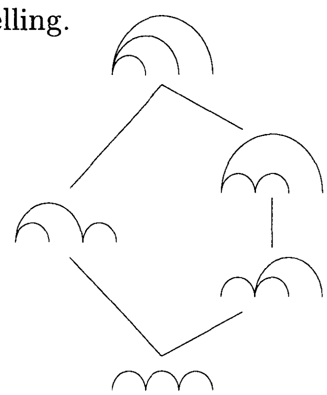

Figure 1-1: A partial order on boolean sublattices in NC_4

For example, the noncrossing partition lattice NC_4 decomposes into five boolean sublattices which are specified by the trees in Figure 1-1. We let the arcs connect the numbers $1, 2, 3, 4$, listed sequentially. One may choose any subset of the arcs in one of these trees to obtain a noncrossing partition; the noncrossing partitions given by a particular tree are thereby arranged into a boolean sublattice of NC_4. In the chain decomposition induced by the partial order on trees in Figure 1-1, the chain $1|2|3|4 \to 13|2|4 \to 1234$ is assigned to the leftmost tree because this chain may be constructed using only the arcs in this tree, but cannot be constructed using only the arcs in the tree below it. The (type A) noncrossing partition lattices have already received considerable scrutiny, but we hope giving such a well-known example will make our strategy of chain decomposition more clear in general

The decompositions given in Chapter 3 are designed to lead easily to flag f-vector formulas; each piece of a chain decomposition will contribute a simple expression to the flag f-vector, and then we sum the results. Recall that the *flag f-vector* of a finite poset of rank n is a function on subsets of $\{1, \ldots, n-1\}$. For each collection $S = \{r_1, \ldots, r_k\} \subseteq \{1, \ldots, n-1\}$, the evaluation of the flag f-vector, denoted by $\alpha_P(S)$, counts the chains passing exactly through ranks $r_1, \ldots, r_k$. For the poset in Figure 1-2, we have $\alpha_P(\emptyset) = 1$, $\alpha_P(\{1\}) = 1$, $\alpha_P(\{2\}) = 2$ and $\alpha_P(\{1, 2\}) = 2$. Ehrenborg

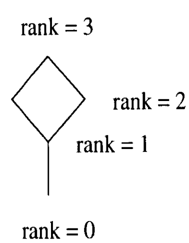

Figure 1-2: A poset with flag f-vector $(1, 1, 2, 2)$

introduced a quasi-symmetric function encoding for the flag f-vector, denoted by F_P, in [Eh, p.9]. Summing over multichains, F_P is defined for nontrivial, ranked posets as

$$F_P = \sum_{\hat{0}=t_0 \leq t_1 \leq \cdots \leq t_{k-1} < t_k = \hat{1}} x_1^{\rho(t_0,t_1)} x_2^{\rho(t_1,t_2)} \ldots x_k^{\rho(t_{k-1},t_k)},$$

where $\rho(x, y)$ is the difference in the ranks of x and y. For the example in Figure 1-2,

$$F_P = \sum_i x_i^3 + \sum_{i<j} x_i x_j^2 + 2 \sum_{i<j} x_i^2 x_j + 2 \sum_{i<j<k} x_i x_j x_k,$$

summing over positive integers. Stanley initiated an investigation of posets for which F_P is a symmetric function in [St4], and in particular observed that F_P is a symmetric function whenever P is locally rank-symmetric. For NC_4, the chains in the earliest tree in Figure 1-1 contribute e_1^3 to F_P while the three intermediate trees each add $e_1 e_2$ to F_P and the last tree contributes e_3 to F_P, so $F_P = e_1^3 + 3e_1 e_2 + e_3$. All four classes of posets to be decomposed in Chapter 3 are locally rank-symmetric, and the different symmetric function bases naturally reflect different topological situations which arise in their order complexes.

The beginning of Chapter 3 makes explicit how the different symmetric function bases may account for the contribution to F_P of different types of regions in the order complex. Because most of the posets we study are shellable and thereby have order complex with homotopy type a wedge of spheres, we are particularly interested in how regions of a sphere may contribute to F_P. Lemma 3.1.1 will be one of our most useful tools. This shows that each skew-Schur function $s_{\lambda/\mu}$ accounts for a patch of a sphere bounded by the restriction to the sphere of hyperplanes that intersect it. The region is thus an intersection of half-spaces restricted to the sphere, and these halfspaces come from inequalities specifying that entries in a semi-standard Young tableau of shape λ/μ must increase weakly in row and strictly in column. Elementary and complete homogeneous symmetric functions are important special cases of Lemma 3.1.1. In contrast, each power-sum symmetric function will account for the restriction to a sphere of a subspace arrangement that intersects it.

The remainder of Chapter 3 provides decompositions into such regions of the order complex and consequent flag f-vector formulas in terms of various symmetric function bases. Shuffle posets of multisets, k-shuffle posets and noncrossing partition lattices for the classical reflection groups decompose into regions which, according to Lemma 3.1.1, each contribute a product of skew-Schur functions to F_P. The

Littlewood-Richardson Rule then implies F_P is Schur-positive for these posets. In the noncrossing partition lattices (of all types) and the traditional shuffle posets, these skew-Schur functions are elementary symmetric functions, while shuffle posets of multisets and k-shuffle posets will involve more general skew-Schur functions of ribbon shape. Subspace arrangements seem to reflect the topology of graded monoid posets much better than patches of spheres, so for these posets we express F_P in terms of power-sum symmetric functions. Stanley conjectured that F_P should be Schur-positive whenever P is locally rank-symmetric and Cohen-Macaulay [St4, p.6], but the graded monoid posets give rise to a counterexample to this conjecture [St5, p.5].

One reason to be interested in when F_P will be a Schur-positive symmetric function is the following observation of Stanley: whenever F_P is a symmetric function, the number of maximal chains in P equals the dimension of the virtual symmetric group representation with F_P as Frobenius characteristic, so there could be a symmetric group action permuting maximal chains which has Frobenius characteristic equalling F_P or ωF_P. Chapter 5 builds on work of Simion and Stanley about symmetric group actions on maximal chains with Frobenius characteristic related to F_P and about chain-labellings known as R^*S-labellings which lead to such actions. There is independent, related work by Rodica Simion and Alina Copeland [Si]. Chapter 5 observes how any R^*S-labelling induces a chain decomposition into regions of the form discussed in Lemma 3.1.1 and shows how to construct a symmetric chain decomposition from any R^*S-labelling. Chapter 5 also gives R^*S-labellings for the noncrossing partition lattices for the classical reflection groups, answering a question raised by Stanley in [St5, p.15].

Chapter 4 gives applications of chain decompositions to shuffle posets of multisets, k-shuffle posets and to some degree also to noncrossing partition lattices for the classical reflection groups. We analyze the topology of order complexes and specialize chain decompositions to 1-chains to obtain symmetric chain decompositions. The chain decompositions of Chapter 3 drew to our attention the existence of M-chains in the various types of shuffle posets and in the noncrossing partition lattices of type A, so we prove supersolvability in each of these cases. For example, the saturated chain $1|2|3|4 \to 12|3|4 \to 123|4 \to 1234$ turns out to be an M-chain in NC_4, essentially because it belongs to each of the boolean sublattices shown in Figure 1-1. Figure 1-3

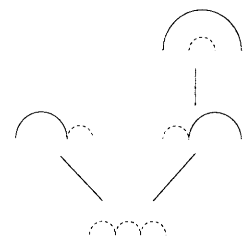

Figure 1-3: A symmetric boolean decomposition for NC_4

gives an example of a symmetric boolean decomposition for NC_4 which yields a symmetric chain decomposition. The noncrossing partitions assigned to any particular tree are those involving all the solid arcs and any subset of the dashed arcs. Notice that the choice of which subset of the dashed arcs to include for each tree above determines a symmetrically placed boolean sublattice. Simple substitutions into our flag f-vector formulas yield rank generating functions, characteristic polynomials and zeta polynomials. In addition, Chapter 4 proves other basic properties for k-shuffle posets, including checking that these are always lattices.

Finally, Chapter 7 gathers together open questions related to our work, many of which are also scattered throughout earlier chapters.

Chapter 2

Definitions of several classes of posets

In Chapter 3 we will give chain decompositions for four classes of posets: shuffle posets of multisets, k-shuffle posets, noncrossing partition lattices for the classical reflection groups and graded monoid posets. Therefore, this chapter briefly reviews the definitions and a few essential features of posets of shuffles, noncrossing partition lattices for the classical reflection groups and graded monoid posets. We also provide a generalization of shuffle posets to shuffling multisets, as defined by Stanley [St7], and introduce the notion of k-shuffle posets, answering a question of Stanley. In addition, we verify that some important structural features of traditional shuffle posets carry over to k-shuffle posets.

2.1 Shuffle posets of multisets

Greene defined and studied posets of shuffles in [Gr, p.191-192], motivated by an idealized model of DNA mutation. Recall that the elements of the shuffle poset $W_{m,n}$ are shuffled words. Let $A_1 = \{a_1, \ldots, a_m\}$ and $A_2 = \{b_1, \ldots, b_n\}$ be two disjoint alphabets, let w_1 be the word $a_1 a_2 \cdots a_m$ and let w_2 be the word $b_1 b_2 \cdots b_n$. We obtain shuffled words by interspersing the letters of w_1 with the letters of w_2, and we denote such shuffled words by $w_1 \sqcup\!\sqcup w_2$. Thus, each possible shuffled word $w_1 \sqcup\!\sqcup w_2$ when restricted to A_1 and A_2, respectively, must satisfy $w_1 \sqcup\!\sqcup w_2|_{A_1} = w_1$ and $w_1 \sqcup\!\sqcup w_2|_{A_2} = w_2$.

The elements of $W_{m,n}$ are all possible subwords of shuffled words $w_1 \sqcup\!\sqcup w_2$. We denote these subwords by $u_1 \sqcup\!\sqcup u_2$ where $u_1 \sqcup\!\sqcup u_2|_{A_1} = u_1$ is a subword of w_1 and $u_1 \sqcup\!\sqcup u_2|_{A_2} = u_2$ is a subword of w_2. The minimal element of $W_{m,n}$ is defined to be the word w_1, the maximal element is defined to be w_2, and there is a covering relation $\mathbf{u} \prec \mathbf{v}$ whenver v is obtained from u by either deleting a letter belonging to A_1 or inserting a letter belonging to A_2 in a way that produces a poset element. It is implicit to this definition that each letter occurs with multiplicity one.

Stanley [St7] generalized this so as to allow repetition of letters in the words to be shuffled. Let w_1 and w_2 be words consisting of letters from disjoint alphabets subject to the constraint that identical letters must always occur consecutively. Let the composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ be the **type** of the word $w = a_1^{\alpha_1} \cdots a_k^{\alpha_k}$. Suppose two words w_1 and w_2 from disjoint alphabets are of type α and β, respectively. These two compositions will determine the shuffle poset of multisets given by w_1 and w_2 up to isomorphism, so we denote this poset by $W_{\alpha,\beta}$. As in traditional shuffle posets, a word w is an element of $W_{\alpha,\beta}$ if w restricted to the alphabet A_1 is a subword of w_1 and w restricted to the alphabest A_2 is a subword of w_2, but with the additional requirement that identical letters must occur consecutively.

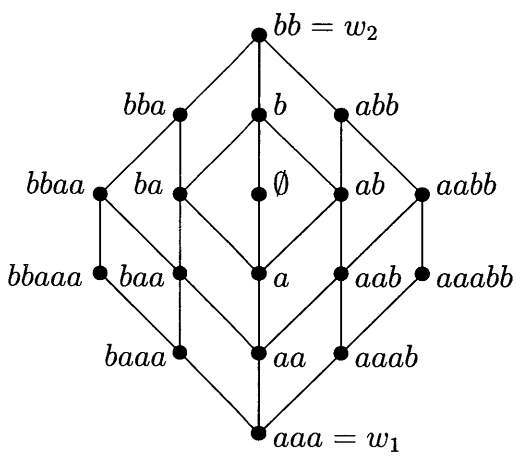

Figure 2-1: The shuffle poset of multisets $W_{3,2}$

For example, if $w_1 = aaab$ and $w_2 = c$, which means $\alpha = (3,1)$ and $\beta = (1)$, then

$aacb$ is a valid poset element while $acab, ba$ and $aaaa$ are not. Similarly to traditional shuffle posets, w_1 is the minimal element, w_2 is the maximal element, and there is a covering relation $u \prec v$ whenever v is obtained from u by either deleting a letter of w_1 or legally inserting a letter of w_2. Figure 2-1 illustrates the poset $W_{3,2}$ with $w_1 = aaa$ and $w_2 = bb$. The traditional shuffle posets are usually denoted $W_{m,n}$, but unfortunately in the notation of shuffle posets of multisets this necessarily becomes $W_{1^m,1^n}$. We note that a different generalization of shuffle posets based on a shuffling operation for lattices has been examined by Doran in [Do].

2.2 k-shuffle posets

We introduce shuffle posets for shuffling k words, answering a question of Stanley. This definition restricts to shuffle posets of multisets which in this context become 2-shuffle posets. The k-shuffle posets are defined in such a way that the i-shuffle poset $W_{\alpha^{(1)},\ldots,\alpha^{(i)}}$ is naturally embedded in the j-shuffle poset $W_{\alpha^{(1)},\ldots,\alpha^{(j)}}$ for $i < j$.

A k-shuffle poset will be specified by k words $w_1, \ldots, w_k$ to be shuffled. We require that the letters of $w_1, \ldots, w_k$ come from disjoint alphabets and insist that identical letters within any particular w_i must occur consecutively. A k-shuffle poset will thus be determined up to isomorphism by an ordered set of k compositions specifying the types of the words $w_1, \ldots, w_k$ to be shuffled. Let $\alpha^{(i)}$ be the composition specifying the type of the word w_i for $1 \leq i \leq k$, and let $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ denote the k-shuffle poset specified $w_1, \ldots, w_k$ are shuffled.

One might expect each k-shuffle poset element to come from shuffling the words $w_1, \ldots, w_k$ and then choosing a subword. However, it is not clear how to partially order such shuffled subwords in a way that will yield a poset (as opposed to a graph with k extreme points or some other structure). Therefore, we define each k-shuffle poset element to be a $(k+1)$-tuple consisting of a subword u_i of the word w_i for $1 \leq i \leq k$ together with a collection of pairwise shuffled words. For each $1 \leq i < j \leq k$, we specify how to shuffle the complement of u_i (viewed as a subword of w_i) with u_j, and the resulting shuffled words comprise this collection of pairwise shuffled words.

In specifying a poset element, we require such a collection of pairwise shuffled words to be consistent, as defined next. Let us denote the complement of u_i within w_i by u_i^c and let $u_i \sqcup\sqcup u_j$ be a word obtained by shuffling u_i and u_j.

Definition 2.2.1 *A collection $\{u_i^c \sqcup\sqcup u_j | i < j\}$ of pairwise shuffled words is* **consistent** *if there is some shuffled word $w_1 \sqcup\sqcup \dots \sqcup\sqcup w_k$ which constains each $u_i^c \sqcup\sqcup u_j$ as a subword.*

Each covering relation will amount to inserting a letter with respect to all "earlier" words and at the same time deleting it with respect to all "later" words, by way of an operation which we therefore call del-sertion.

Definition 2.2.2 *If C is a consistent collection of pairwise shuffled words $\{u_i^c \sqcup\sqcup u_j | i < j\}$ and b is a letter belonging to some u_l, then we* **del-sert** *b by deleting b from each copy of u_l^c and at the same time inserting b in each copy of u_l in C.*

In order for v to cover u, v must be obtained from u by del-serting a letter which belongs to some u_i in such a way that the collections of pairwise shuffled words associated to u and v are consistent in the following sense.

Definition 2.2.3 *Let $\{u_i^c \sqcup\sqcup u_j | i < j\}$ and $\{v_i^c \sqcup\sqcup v_j | i < j\}$ be the collections of pairwise shuffled words in two poset elements u and v, respectively. The poset elements u and v are* **consistent** *if there exists some shuffled word $w_1 \sqcup\sqcup \dots \sqcup\sqcup w_k$ which simultaneously contains each $u_i^c \sqcup\sqcup u_j$ and each $v_i^c \sqcup\sqcup v_j$ as a subword.*

In summary, we have the following definition.

Definition A *The* **k-shuffle poset** *$W_{\alpha^{(1)},\dots,\alpha^{(k)}}$ is given by the following elements and covering relations.*

1. *Let u be an element of $W_{\alpha^{(1)},\dots,\alpha^{(k)}}$ if $u = (u_1, \dots, u_k, \{u_i^c \sqcup\sqcup u_j | i < j\})$, where u_i is a subword of w_i for $1 \leq i \leq k$ and the collection of pairwise shuffled words $\{u_i^c \sqcup\sqcup u_j | i < j\}$ is consistent.*

2. *Let* $u \prec v$ *if* v *is obtained from* u *by del-serting a letter from some* u_i *and if* v *is consistent with* u.

The consistency requirement on covering relations $u \prec v$ is automatic for $k = 2$, but a necessary assumption for larger k. For example, let $w_1 = 1, w_2 = b, w_3 = C$ and let e denote the empty word. If $u = (e, e, C, \{1, 1C, Cb\}$ and $v = (e, b, C, \{b1, 1C, C\}$, then one might hope to obtain v from u by del-serting b, but this is not allowed because u and v are not consistent. See Figure 2-2 for the entire 3-shuffle poset in this case. We label poset elements with the sets of shuffled words $\{u_1^c \sqcup\!\sqcup u_2, u_1^c \sqcup\!\sqcup u_3, u_2^c \sqcup\!\sqcup u_3\}$. Proposition 2.2.2 will show that edge consistency implies consistency of all poset

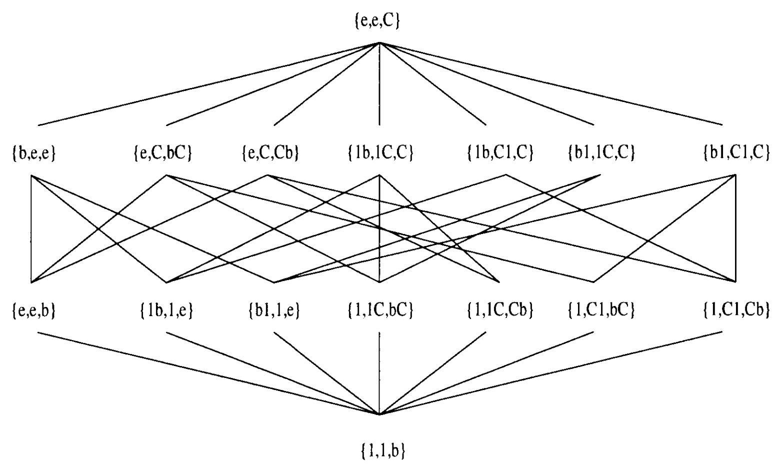

Figure 2-2: A k-shuffle poset

chains.

Next we give a more complicated constructive definition for k-shuffle posets and check its equivalence to Definition A. This will be useful in Chapter 3. First note that each possible shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ gives rise to a product of chains subposet of the form C_α where α is the composition obtained by taking the union of the compositions $\alpha^{(1)}, \ldots, \alpha^{(k)}$ for w_i of type $\alpha^{(i)}$. For each w, we denote by P_w the product of chains subposet of those elements of $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ which are consistent with $w = w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$.

Each shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ may equivalently be represented by a consistent collection of pairwise shuffled words $\{w_i \sqcup\!\sqcup w_j | i < j\}$. If we let e_j denote the empty word considered as a subword of w_j, then each product of chains will have a collection $\{w_i \sqcup\!\sqcup e_j | i < j\}$ as its minimal element. Each covering relation will amount to del-serting a letter in the unique way that is consistent with the shuffled word specifying the product of chains. Thus, the labels on covering relations in Definition B may be viewed as the letters to be del-serted.

Definition B *The k-shuffle poset $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ is constructed as follows.*

1. *Let $\hat{0} = \{w_i \sqcup\!\sqcup e_j | i < j\}$.*

2. *If $w = w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ is a shuffling of $w_1, \ldots, w_k$, then label covering relations within the product of chains P_w by letters in $w_1, \ldots, w_k$ in the natural way. Namely, if $u \prec v$ and v differs from u in the coordinate in position $l(\alpha^{(1)}) + \cdots + l(\alpha^{(i)}) + j$ in P_w, then the covering relation $u \prec v$ is labelled with the jth distinct letter in w_{i+1}. Label v with the collection of pairwise shuffled words obtained by del-serting the label of the edge (u, v) into the collection of pairwise shuffled words for u.*

3. *Glue together two elements from distinct products of chains P_w and $P_{w'}$ if they are specified by identical collections of pairwise shuffled words. Glue together two covering relations $u \prec v$ and $u' \prec v'$ if u is glued to u' and v is glued to v'.*

Proposition 2.2.1 *Definitions A and B are equivalent.*

PROOF. In Definition B, saturated chains are given by the order in which labels occur and by the collection of shuffled words $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ with which they are consistent; Proposition 2.2.2 will ensure that every saturated chain is consistent with at least one shuffled word. The label order in a chain given by Definition B determines del-sertion order in a Definition A chain, and the positions in which letters are del-serted is completely determined by the collection of shuffled words with which the chain is consistent. Identification of elements from distinct products of chains in Definition B is tantamount to deletion before insertion in Definition A. This map of chains induces an order-preserving bijection between poset elements. $\square$

We conclude with a (somewhat technical) check that each poset chain is consistent with at least one shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$.

Definition 2.2.4 *A chain contains a* **loop** $a_1, \dots, a_m$ *if the letters* $a_1, \dots, a_m$ *all occur in pairwise shuffled words in the chain in such a way that each* a_i *precedes* a_{i+1} *in some pairwise shuffled word in the chain, and if* a_m *also precedes* a_1 *in some chain element.*

If there is an inconsistency in a saturated chain, there must be a loop. If a letter a_i is del-serted in a covering relation $u \prec v$, then let $t(a_i)$ be the rank of v, since in some sense this as the time at which a_i is del-serted in travelling from $\hat{0}$ to $\hat{1}$. Let $w(a_i)$ be the index of the word to which a_i belongs, namely if $a_i \in w_j$ for $1 \le j \le k$ then $w(a_i) = j$. We say that one letter precedes another at $t(a_i)$ if this is true in any of the pairwise shuffled words in either u or v.

Proposition 2.2.2 *Every chain in a k-shuffle poset is consistent with at least one way of shuffling* $w_1, \dots, w_k$.

PROOF. It suffices to verify this for saturated chains. This will amount to showing that whenever a saturated chain has an inconsistency, there is some inconsistent edge in the chain. To simplify notation, we assume that each letter occurs with multiplicity one because the general case is essentially the same.

Suppose the letters $a_1, \dots, a_m$ form a loop, but every edge is consistent. Without loss of generality, we may assume $t(a_m) > t(a_1)$. This implies $w(a_m) \ge w(a_1)$ since a_1 and a_m have comparable positions at some time. Similarly, note that $t(a_i) < t(a_{i+1})$ implies $w(a_i) \le w(a_{i+1})$ and $t(a_i) > t(a_{i+1})$ implies $w(a_i) \ge w(a_{i+1})$ for $1 \le i \le m-1$. If $t(a_i) > t(a_m)$ and $w(a_i) \ge w(a_m)$ for some $1 < i < m$, then we get a loop a_1, a_i, a_m at time $t(a_i)$. Similarly, we cannot have $t(a_i) < t(a_1)$ and $w(a_i) \le w(a_1)$ for $1 < i < m$. In particular, $t(a_2) > t(a_1)$ and $t(a_{m-1}) < t(a_m)$.

Let us first consider the case $t(a_m) = t(a_1) + 1$. This implies $t(a_2) > t(a_m)$ and $w(a_1) \le w(a_2) < w(a_m)$. By the same reasoning, $t(a_{m-1}) < t(a_1)$ and $w(a_1) < w(a_{m-1}) \le w(a_m)$. In some sense, the Intermediate Value Theorem then implies the existence of some $1 < i < m$ such that $t(a_i) > t(a_m) > t(a_1) > t(a_{i+1})$. We must have $w(a_1) \leq w(a_{i+1}) \leq w(a_i) \leq w(a_m)$; otherwise we would have $w(a_{i+1}) < w(a_1)$ or $w(a_i) > w(a_m)$, either of which would lead to a loop of size three. However, the inequalities $w(a_1) \leq w(a_{i+1}) \leq w(a_i) \leq w(a_m)$ imply a loop a_1, a_i, a_{i+1}, a_m, giving an inconsistent edge at $t(a_1)$.

When $t(a_m) > t(a_1)+1$, the same argument applies unless there is some $1 < i < m$ such that $t(a_1) < t(a_i) < t(a_m)$. If so, $w(a_i) > w(a_m)$ or $w(a_i) < w(a_1)$. Without loss of generality, assume the former. Since $w(a_{m-1}) < w(a_m)$, there exists some $j \geq i$ such that $w(a_j) > w(a_m) > w(a_{j+1})$, while $t(a_j) > t(a_1)$ and $t(a_{j+1}) < t(a_m)$. This gives rise to a loop a_1, a_j, a_{j+1}, a_m which yields an inconsistency at $t(a_j)$. Hence, there must always be an inconsistent edge □

2.3 Noncrossing partition lattices for the classical reflection groups

The noncrossing partition lattice NC_n is a meet-sublattice of the lattice Π_n of partitions of $\{1, \ldots, n\}$ ordered by refinement. We restrict to those partitions which are **noncrossing** in the following sense. If $a, c \in C_1$ and $b, d \in C_2$ for $a < b < c < d$, then $C_1 = C_2$. We may write the numbers $1, \ldots, n$ sequentially on a number line and draw arcs above the number line connecting any two numbers in the same component of a partition; if the arcs may be drawn in such a way that removing every pair of arcs that cross each other does not change the partition, then the partition is said to be noncrossing. The point is that if we take the convex hull of all the arcs between the elements of any particular block, and we do this for every block, then in a noncrossing partition these convex hulls need not intersect each other.

Reiner generalizes NC_n to the classical reflection groups in [Re]. He bases his definition on the interpretation of the partition lattice as the intersection lattice of the type A Coxeter arrangement; he takes the intersection lattices for the other Coxeter hyperplane arrangements and defines what it means for an element to be noncrossing. The hyperplanes in the type B Coxeter arrangement are of the form $x_i \pm x_j = 0$ for $1 \leq i \leq j \leq n$. Hence, we partition $\pm 1, \pm 2, \ldots, \pm n$ in such a way that i is in the same component as j if and only if $-i$ is in the same component as $-j$. To define type B noncrossing partitions, we place the numbers $1, 2, \ldots, n, -1, -2, \ldots, -n$ clockwise about a circle so that i is exactly opposite $-i$. Let $i \lhd j$ if i is at most 180 degrees counterclockwise from j, noting that this is not a transitive relation. A type B partition is **noncrossing** if $a \lhd b \lhd c \lhd d$ and $a \lhd d$ for $a, c \in C_1$ and $b, d \in C_2$ implies $C_1 = C_2$. This is analogous to the condition for type A because we may draw arcs inside the circle connecting the elements of each component in such a way that the partition remains unchanged if arcs that cross each other are deleted.

Note that the condition that $i, j \in C$ if and only if $-i, -j \in -C$ amounts to 180 degree rotational symmetry. For each component C there will be a component $-C$ such that $j \in C$ if and only if $-j \in -C$. In particular, one component may satisfy $C = -C$. When such a component exists, this is called the 0-component and is denoted by C_0. The noncrossing property precludes the existence of more than one 0-component.

The **type B noncrossing partition lattice** is the lattice of type B noncrossing partitions ordered by refinement. The **type D noncrossing partition lattice** is the restriction to noncrossing partitions with $C_0 \neq \{\pm i\}$ for $1 \leq i \leq n$, since we no longer allow hyperplanes of the form $x_i = 0$. The **interpolating BD noncrossing partition lattices** are given by choosing a subset $S \subseteq \{1, \ldots, n\}$ and forbidding noncrossing partitions with $C_0 = \{\pm i\}$ for $i \in S$.

2.4 Graded monoid posets

Monoid posets arise in the work of Peeva, Reiner and Sturmfels [PRS] as a tool for studying free resolutions of monomial ideals and in particular for computing $\mathrm{Tor}^{\mathrm{k}[\Lambda]}(\mathrm{k}, \mathrm{k})$, where Λ is a finitely generated submonoid of $\mathbb{N}^d$. Shellings are constructed from quadratic Gröbner bases in [PRS], and we will obtain chain decompositions from Gröbner bases in a somewhat similar fashion; therefore, we define monoid posets not only in terms of vector sums, but also (equivalently) in terms of products of monomials. Each monoid generator $(n_1, \ldots, n_d) \in \mathbb{N}^d$ naturally corresponds to a monomial $z_1^{n_1} \ldots z_d^{n_d}$ with vector addition replaced by multiplication of monomials.

Any finite collection C of vectors in $\mathbb{N}^d$ gives rise to a **monoid poset** denoted by P as follows: let $\hat{0}$ be the 0-vector, let $u \in P$ if u is a sum of elements in C, and let $u \prec v$ if $v = u + w$ for some $w \in C$. Following [PRS], let $k[\Lambda]$ be the ring generated by the monomials corresponding to the generators of a monoid Λ. Assume that Λ is generated by $\mathbf{a_1}, \ldots, \mathbf{a_n}$ where $\mathbf{a}_i = (a_{i1}, \ldots, a_{id}) \in \mathbb{N}^d$. Note that $k[\Lambda] \cong k[x_1, \ldots, x_n]/I_\Lambda$ where I_Λ is the kernel of the map

$$\phi : k[x_1, \ldots, x_n] \to k[z_1, \ldots, z_d]$$

which sends x_i to $z_1^{a_{i1}} \ldots z_d^{a_{id}}$. Thus, I_Λ is the toric ideal of syzygies among the images under ϕ of the monomials $x_1, \ldots, x_n$. In this language, the monomials in $k[x_1, \ldots, x_n]/I_\Lambda$ form an (infinite) monoid poset. The minimal element is the multiplicative identity and monomials in $k[x_1, \ldots, x_n]/I_\Lambda$ are partially ordered by divisibility.

If the generators of a monoid all lie in a hyperplane, then the monoid is **graded** and it makes sense to speak of the flag f-vector of its monoid poset, so these are the monoid posets we will consider in Chapter 3. It is observed in [PRS] that every interval $[u, v]$ in a graded monoid poset is isomorphic to an interval of the form $[\hat{0}, w]$, obtained by subtracting u from each monoid element in the interval, so it suffices to study intervals of the form $[\hat{0}, u]$.

Chapter 3

Chain decompositions and formulas for the flag f-vector

This chapter provides explicit chain decompositions for several classes of posets and derives from these flag f-vector formulas in terms of symmetric functions. Recall our basic plan of attack of decomposing the space of chains in a poset by specifying a collection of subposets, partially ordering these subposets and then assigning each poset chain to the earliest of these subposets which contains it. This will give a chain decomposition, provided that every chain belongs to at least one of these subposets and that when a chain belongs to more than one, then one of these subposets containing it comes earlier than all the others. For the most part, the posets we study are shellable, and our chain decompositions will break their order complexes into two types of regions:

1. A patch of a sphere where the patch is bounded by the restriction to the sphere of an arrangement of hyperplanes that intersect it.

2. The restriction to a sphere of a central subspace arrangement.

Even for the posets we study which are not shellable, we still decompose their order complexes into these two types of regions. In Section 3.1, we show how products of skew-Schur functions account for the contribution to F_P of the chains comprising a region of the former type; on the other hand, power-sum symmetric functions will naturally handle the latter type of region. Note that elementary symmetric functions, complete homogeneous symmetric functions and ordinary Schur functions all fall into the category of products of skew-Schur functions.

We will decompose the order complexes of shuffle posets of multisets, k-shuffle posets and noncrossing partition lattices for the classical reflection groups into patches of spheres which each contribute to F_P a product of skew-Schur functions as specified by Lemma 3.1.1. Together with the Littlewood-Richardson Rule, these decompositions will thereby imply Schur-positivity of F_P in these cases. In contrast, graded monoid posets are not always Schur-positive, even assuming they are shellable [St5, p.5]. We will express F_P in terms of power-sum symmetric functions for a special class of graded monoid posets, and then use plethystic substitution of power-sum symmetric functions into complete homogeneous symmetric functions to handle the general case.

3.1 Topological interpretations for the symmetric function bases

Our choice of which symmetric function basis to use for a particular class of posets depends very much on the nature of the regions into which its order complex decomposes topologically, so let us begin by showing how the different symmetric function bases naturally reflect different scenarios.

First recall the quasi-symmetric function encoding

$$F_P = \sum_{\hat{0}=t_0 \le t_1 \le \cdots \le t_{k-1} < t_k = \hat{1}} x_1^{\rho(t_0,t_1)} x_2^{\rho(t_1,t_2)} \cdots x_k^{\rho(t_{k-1},t_k)}$$

for the flag f-vector of a finite, nontrivial ranked poset with $\hat{0}$ and $\hat{1}$, as introduced by Ehrenborg in [Eh, p.9-10]. In this expression, $\rho(x, y)$ denotes the difference in the ranks of x and y, and the sum is over all multichains of any length, as long as they include at least one copy of $\hat{0}$ and exactly one copy of $\hat{1}$. Stanley showed that F_P is a symmetric function whenever P is locally rank-symmetric in [St4, p.4-5]. Ehrenborg

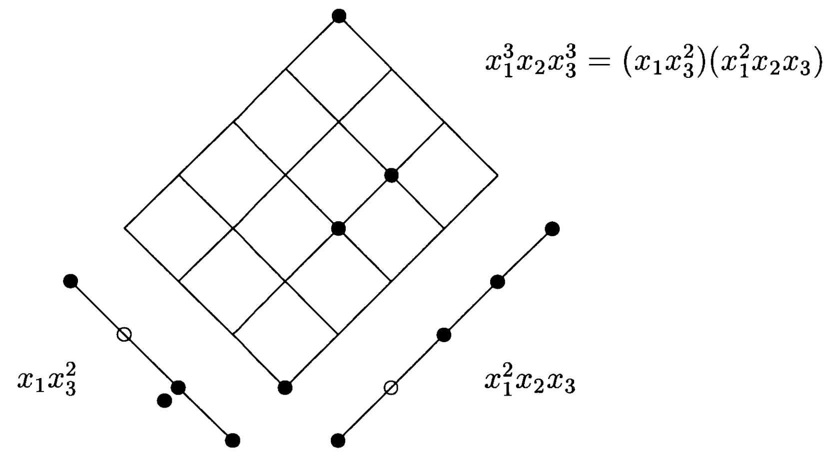

Figure 3-1: $F_{P\times Q} = F_P F_Q$

observed in [Eh, p.10] that $F_{P\times Q} = F_P F_Q$. Figure 3-1 gives an example of how a multichain in P together with a multichain in Q give rise to a multichain in $P \times Q$ and their contributions to F_P are multiplied; when such a pair of multichains in P and Q differ in length, one must first augment the shorter one with copies of $\hat{1}$ before pairing up multichain elements. Ehrenborg also noticed that $F_P = h_n$ for P a chain of rank n, since each possible monomial of degree n is given by a single multichain in C_{n+1}. Combining these facts shows that $F_P = h_\lambda$ when P is the product of chains $C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$.

The collection of multichains in a chain C_{n+1} which never jump in rank by more than one between consecutive multichain elements contributes e_n to F_P. Simply note that e_n is the sum of all monomials of degree n in which variables may never repeat. Invoking the multiplicative nature of F_P, observe that e_λ accounts exactly for the multichains in $C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$ which jump by at most one in each coordinate between consecutive multichain elements. For instance, the multichain in Figure 3-1 would not be allowed.

Simion and Stanley expressed this relationship between h_λ and e_λ in terms of labellings known as R^*S-labellings in [SS, p.16-21]. Chapter 5 will relate these labellings to our study of chain decompositions. However, the symmetric functions h_λ

and e_λ may also be considered as products of Schur functions. We next disuss how arbitrary products of Schur functions and more generally of skew-Schur functions account for collections of poset chains in F_P. For each skew-shape λ/μ, Lemma 3.1.1 will specify a collection of chains within a boolean lattice which contributes $s_{\lambda/\mu}$ to F_P. Each such collection will constitute a patch of a sphere which is the geometric realization of the order complex. Later sections in this chapter will decompose the order complexes of several classes of posets into such regions. We note that alternatively the order complex of the boolean lattice B_n may be entirely decomposed into regions accounted for by Schur functions using the Robinson-Schensted correspondence. However, Lemma 3.1.1 relies on a way of assigning chains in a boolean lattice to hyperplanes and to open and closed half-spaces, as described next.

The elements of the boolean lattice B_n naturally correspond to the subsets of a set $\{a_1, \ldots, a_n\}$, and each multichain $\hat{0} = v_0 \leq v_1 \leq \cdots \leq v_{k-1} < v_k = \hat{1}$ thereby gives rise to a string of inclusions $\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{k-1} \subset S_k = \{a_1, \ldots, a_n\}$. Let V be a real vector space with coordinates $a_1, \ldots, a_n$. We assign each multichain in B_n to an intersection of hyperplanes and open half-spaces restricted to the hyperplane $\sum_{i=1}^n a_i = 0$ in V as follows. Partition $\{a_1, \ldots, a_n\}$ into blocks $B_1, \ldots, B_k$ by letting $S_i \setminus S_{i-1} = B_i$ for $1 \leq i \leq k$. A multichain is then assigned to the intersection of all the hyperplanes $a_j = a_{j'}$ such that $a_j, a_{j'} \in B_i$ for $1 \leq i \leq k$ with all the open half-spaces $a_j < a_{j'}$ such that $a_j \in B_i$ and $a_{j'} \in B_{i'}$ for $1 \leq i < i' \leq k$.

The hyperplane arrangment given by the hyperplanes $a_i = a_j$ for $1 \leq i < j \leq n$ decomposes V into cones which are bounded by these hyperplanes. Let us restrict this decomposition of V to the $(n-2)$-sphere which is obtained by taking the slice of the unit sphere $\sum_{i=1}^n a_i^2 = 1$ which intersects the hyperplane $\sum_{i=1}^n a_i = 0$. The hyperplane arrangement thereby specify a triangulation of the $(n-2)$-sphere which by definition consists of the same simplices as the order complex for B_n. Each i-chain in B_n gives rise to an $(i-1)$-face in its order complex, and our assignment sends each chain to a region of dimension $i-1$ which indeed corresponds to the $(i-1)$-face of the order complex. This discussion is informed by a similar point of view in [HRW, p.5-11].

We may similarly view elements of B_n as the subsets of a set $\{a_{i,j} | j \leq \lambda_i\}$ given by any partition $\lambda \vdash n$. In this context, h_λ accounts for the closed patch of the $(n-2)$-sphere satisfying the weak inequalities

$$\begin{aligned} a_{1,1} &\leq a_{1,2} \leq \cdots \leq a_{1,\lambda_1} \\ a_{2,1} &\leq a_{2,2} \leq \cdots \leq a_{2,\lambda_2} \\ &\cdots \\ a_{k,1} &\leq a_{k,2} \leq \cdots \leq a_{k,\lambda_k} \end{aligned}$$

while e_λ accounts for the open patch of the order complex given by the system of strict inequalities

$$\begin{aligned} a_{1,1} &< a_{1,2} < \cdots < a_{1,\lambda_1} \\ a_{2,1} &< a_{2,2} < \cdots < a_{2,\lambda_2} \\ &\cdots \\ a_{k,1} &< a_{k,2} < \cdots < a_{k,\lambda_k}. \end{aligned}$$

More generally, we may define a system of inequalities based on the fact that the entries in semi-standard Young tableaux of shape λ/μ must increase weakly in rows and strictly in columns. Lemma 3.1.1 will show that a region specified by such a system of constraints contributes $s_{\lambda/\mu}$ to F_P. In this case, let $S = \{a_{i,j} | \mu_i < j \leq \lambda_i\}$. Whenever the Young diagram of shape λ/μ includes a pair of neighboring boxes in positions (i,j) and $(i+1,j)$, we introduce a weak inequality $a_{i,j} \leq a_{i+1,j}$, and for each pair of neighboring boxes in positions (i,j) and $(i,j+1)$, we establish a strict inequality $a_{i,j} < a_{i+1,j}$. Let $n = l - k$ for $\mu \subseteq \lambda$ satisfying $\mu \vdash k$ and $\lambda \vdash l$.

Lemma 3.1.1 *The collection of multichains in a boolean lattice B_n satisfying the constraints described above for the skew-shape λ/μ contributes the skew-Schur function $s_{\lambda/\mu}$ to F_P.*

PROOF. This is a direct consequence of the combinatorial definition of skew-Schur function. Considering $s_{\lambda/\mu}$ as a sum over the semi-standard Young tableaux of shape λ/μ, we need only show that each such tableau giving rise to a monomial of content ν corresponds to a multichain in the bounded region which contributes x^{ν} to F_P, and that this correspondence is a bijection. The bijection comes from placing the number d in the box at position (i, j) in a SSYT of shape λ/μ for $a_{i,j} \in S_d \setminus S_{d-1}$ in the corresponding multichain $\emptyset = S_0 \subseteq \cdots \subseteq S_{k-1} \subset S_k = \{a_{i,j} | \mu_i < j \leq \lambda_i\}$. The constraints on multichains in a region are designed to correspond to the constraints on semi-standard Young tableaux entries so that legal multichains are mapped to legal SSYT. The monomials will agree because $|S_d| - |S_{d-1}|$ for a multichain will be the number of boxes containing d in the corresponding SSYT. $\square$

Example 3.1.1 *As an example, the semi-standard Young tableau shown in Figure 3-2 corresponds to the multichain* $\emptyset \subseteq \{a_{2,2}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}, a_{1,4}\} \subseteq$

			3
	1	5	6
2	2	6	

Figure 3-2: A semi-standard Young tableau of content $(1, 2, 1, 0, 1, 2)$

$\{a_{2,2}, a_{3,1}, a_{3,2}, a_{1,4}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}, a_{1,4}, a_{2,3}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}, a_{1,4}, a_{2,3}, a_{2,4}, a_{3,3}\}$. *Both contribute monomials* $x_1 x_2^2 x_3 x_5 x_6^2$ *to their respective manifestations of the skew-Schur function* $s_{(4,4,3)/(3,1)}$.

Elementary symmetric functions come up in applications of Lemma 3.1.1 to non-crossing partition lattices while more general products of skew-Schur functions of ribbon shape will arise in our analysis of shuffle posets of multisets and more general k-shuffle posets. Next, we show how power-sum symmetric functions naturally account for the multichains assigned to the subspaces in a subarrangement of a Coxeter hyperplane arrangement of type A. This will be useful for our study of graded monoid posets.

Recall the interpretation of the lattice of partitions of $\{1, \ldots, n\}$ as the intersection lattice of the type A Coxeter arrangement. This arrangement is given by the hyperplanes $a_i = a_j$ for $1 \leq i < j \leq n$, and a subspace yields a partition by assigning i and j to the same partition block whenever the corresponding subspace satisfies $a_i = a_j$. By convention, the **type** of a subspace is the type of the corresponding partition in Π_n, namely the partition of n recording the block sizes.

Note that the multichains in the boolean lattice B_n which (when considered in the order complex) lie in a particular subspace of type λ contribute p_λ to F_P. To see this, recall that $x_1^{n_1} \dots x_j^{n_j}$ accounts for the contribution to F_P of a multichain with jumps of sizes $n_1, \dots, n_j$. Since $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$ and $p_r = \sum_{i>0} x_i^r$, we note that p_λ will account for all multichains with a particular collection of jumps of sizes $\lambda_1, \dots, \lambda_k$ taken in any order or with any of these jumps merged together. These are exactly the multichains assigned to a subspace of type λ, because then requiring coordinates to be equal amounts to insisting they occur in the same jump in a multichain. Figure 3-3

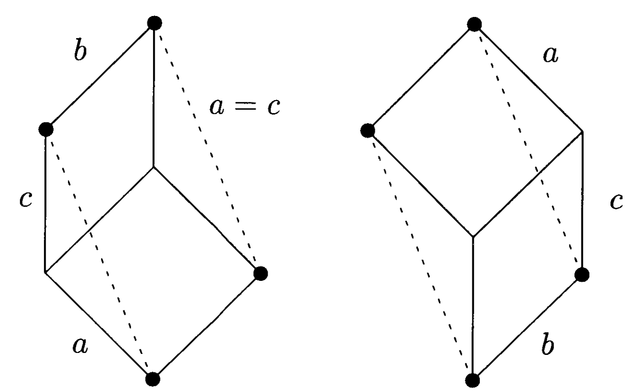

Figure 3-3: Two sublattices of B_3 with intersection contributing $p_1 p_2$ to F_P

gives an example of two subposets whose images in the order complex are half-spheres; these two patches which each cover half the sphere overlap in the restriction to the sphere of a subspace of type $(2, 1)$ given by chains assigned to the hyperplane $a = c$.

This topological picture together with Lemma 3.1.1 gives one way of encoding relationships between power-sum symmetric functions and other symmetric function bases. Power-sum symmetric functions arise in our flag f-vector formulas for graded monoid posets in what is known as the squarefree case, because once we partially order the cycles in a homology basis, the intersection of each sphere in the homology basis with the union of earlier ones will be the restriction to that sphere of a subspace arrangement that is contained in a type A Coxeter arrangement. Sieve methods allow us to give a formula for F_P using Möbius functions of intersection lattices to keep track of overlap. For more general graded monoid posets, we will use plethystic substitution of power-sum symmetric functions into complete symmetric functions, because we will have strings of weak inequalities on identical jumps of arbitrary size in the same fashion that we had weak inequalities on identical jumps of size one in our discussion of complete symmetric functions.

3.2 Shuffle posets of multisets

We begin with a simple, but hopefully suggestive example.

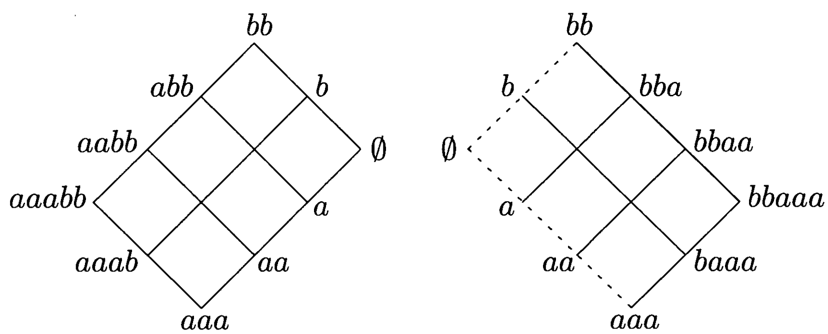

Figure 3-4: A chain decomposition for $W_{3,2}$

Let $w_1 = aaa$ and $w_2 = bb$, as in Figure 3-4. We decompose the space of maximal chains into two pieces and account for the contribution of each separately to F_P. Chains in which each element is a subword of $w_1 = aaa$ followed by a subword of $w_2 = bb$ are exactly the chains in the product of a 3-chain with a 4-chain, shown on the left in Figure 3-4. Hence, these contribute $h_3 h_2 = s_{\square\square\square}\, s_{\square\square}$ to F_P.

The chains with the letter b occuring immediately before the letter a in some element of the chain give another copy of $h_3 h_2$ for the product of chains from subwords of bb followed by subwords of aaa, shown on the right in Figure 3-4. We must subtract for overlap, which means chains in which a and b never appear together. These are the chains contained in the maximal chain $aaa < aa < a < \emptyset < b < bb$, so we subtract

h_5 to obtain $F_P = h_3h_2 + (h_3h_2 - h_{3+2}) = s_{(3)}\, s_{(2)} + s_{(4,3)/(2)}$ for $P = W_{3,2}$.

We may embed either piece of the decomposition into a boolean lattice with atoms a_1, a_2, a_3, b_1, b_2 by imposing the constraints $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2$. For the latter piece of the decomposition, we also need the strict inequality $b_1 < a_3$ to represent the requirement that not all three copies of a be deleted before the first copy of b is inserted.

Let m and n be the lengths of α and β, respectively, and let γ be the composition obtained by concatenating the compositions α and β. Our chain decomposition for $W_{\alpha,\beta}$ has four steps:

1. Break a shuffle poset of multisets into overlapping products of chains. Each shuffled word $w = w_1 \sqcup\!\sqcup\, w_2$ gives rise to such a subposet P_w consisting of all the subwords of $w_1 \sqcup\!\sqcup\, w_2$. The poset P_w is a product of chains C_γ.

2. Partially order the P_w by partially ordering the shuffled words specifying them. Using one-line notation, our partial order on shuffled words is the interval in the weak order from $(n, \ldots, 1, m+n, \ldots, n+1)$ to the reverse permutation $(m+n, \ldots, 1)$. (This goes against the usual convention of swapping adjacent values in weak order covering relations; in studying shuffled words, it seems more natural to swap adjacent positions which amounts to taking the weak order interval on inverse permutations.)

3. Assign each poset chain to the earliest product of chains containing it. Note that each poset chain belongs to some P_w. Also, the choice of which P_w containing a chain comes earliest will be well-defined because of what are known as interface pairs.

4. Compute F_P for each piece of the decomposition and sum the results.

Expressing the decomposition in terms of inequalities as in the above example will lead by way of Lemma 3.1.1 to the following formula for F_P in terms of skew-Schur functions.

Theorem 3.2.1 *The flag f-vector F_P for $P = W_{\alpha,\beta}$ is*

$$\sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j} s_{(\alpha_{a_i}+\beta_{b_i}-1,\beta_{b_i})/(\beta_{b_i}-1)}\right)\left(\prod_{i\notin\{a_1,\ldots,a_j\}} s_{\alpha_i}\right)\left(\prod_{i\notin\{b_1,\ldots,b_j\}} s_{\beta_i}\right).$$

Equivalently, F_P may be expressed in terms of complete symmetric functions as

$$\sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j}(h_{\alpha_{a_i}}h_{\beta_{b_i}} - h_{\alpha_{a_i}+\beta_{b_i}})\right)\left(\prod_{\substack{i\notin\{a_1,\ldots,a_j\}\\ 1\le i\le l(\alpha)}} h_{\alpha_i}\right)\left(\prod_{\substack{i\notin\{b_1,\ldots,b_j\}\\ 1\le i\le l(\beta)}} h_{\beta_i}\right).$$

In the special case of traditional shuffle posets, namely $W_{1^m,1^n}$, this becomes

$$F_P = \sum_{j=0}^{\min(m,n)} \binom{m}{j}\binom{n}{j} e_2^j e_1^{n+m-2j},$$

so we recover a formula of [SS, p.21] for traditional shuffle posets. Simply substitute e_2 for $h_1h_1 - h_2$ and e_1 for h_1 above.

Corollary 3.2.1 *F_P is Schur-positive for shuffle posets of multisets.*

PROOF. Recall that skew-Schur functions are Schur-positive and that the Littlewood-Richardson Rule asserts that products of Schur functions are also Schur-positive. □

Let us give a more substantial example before verifying the flag f-vector formula.

Example. *Let $w_1 = 12222$ and $w_2 = aabbb$. On the left side in Figure 3-5, we partially order the shuffled words specifying the product of chain sublattices. On the right, we record their corresponding contributions to F_P, so in this case F_P is the sum of these complete symmetric functions.*

To prove Theorem 3.2.1 we will specify constraints on the collection of chains in each piece of the decomposition. To this end, we introduce what is known as the interface of a chain. This will be a direct generalization of Greene's notion of the interface of a poset element in [Gr, p.195-196].

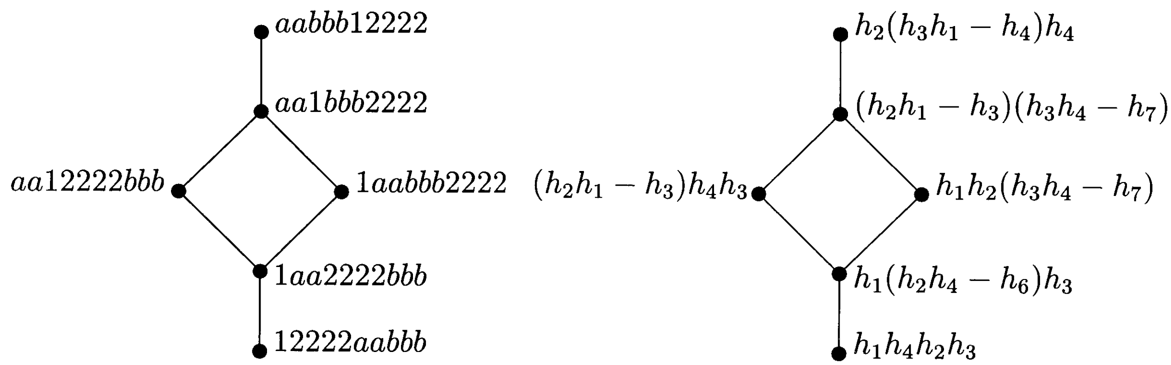

Figure 3-5: Summing contribution to F_P

Recall that the **interface** of a traditional shuffle poset element $u_1 \sqcup\!\sqcup u_2$ is the collection of pairs of letters (a, b) such that a belongs to w_2, b belongs to w_1, and a immediately precedes b in the shuffled word $u_1 \sqcup\!\sqcup u_2$, so the interface determines the degree to which u_1 and u_2 are shuffled. Although letters may occur with multiplicity in shuffle posets of multisets, we refer to pairs of letters as belonging to the interface when we actually mean pairs of classes of identical letters, since identical letters always occur consecutively.

Definition 3.2.1 *The* **interface** *of a chain is obtained by taking the union of the interfaces of all chain elements, then eliminating those ordered pairs which are pre-empted by other "more shuffled" pairs arising elsewhere in the chain. One pair pre-empts another if it consists of the same letter or a later letter of w_2 and the same letter or an earlier letter of w_1.*

Following [Gr, p.195-196], letters not occuring in the interface of a chain comprise the **residue** of the chain. As we mentioned before, each possible shuffled word $w_1 \sqcup\!\sqcup w_2$ yields a product of chains sublattice consisting of all subwords of this shuffled word. Partitioning chains according to their interface amounts to assigning each chain to the product of chains containing it which comes earliest in this partial order, namely the one specified by the least shuffled word.

Proof of Theorem 3.2.1 Each summand in Formula 1 accounts for chains with a particular interface, specified by sets $\{a_1, \ldots, a_j\} \subseteq \{1, \ldots, l(\alpha)\}$ and $\{b_1, \ldots, b_j\} \subseteq \{1, \ldots, l(\beta)\}$ which index the distinct letters from w_1 and w_2, respectively. The collection of multichains with this particular interface will be the chains in the product of chains P_w where w has exactly this interface and the chains satisfy the following constraints. When a letter r_i occurs with multiplicity k, we denote the k copies by $r_{i_1}, \ldots, r_{i_k}$ and impose the constraints $r_{i_1} \leq \cdots \leq r_{i_k}$ on multichains, so as to embed P_w in a boolean lattice. Furthermore, for each interface pair (a_i, b_i) of letters occuring with multiplicities m and n, respectively, there is a constraint $b_{i_1} < a_{i_m}$ to reflect the fact that the first copy of b_i must be inserted before the last copy of a_i is deleted; otherwise, the multichain be consistent with some earlier w' not containing this interface pair. Thus, each interface pair gives rise to a set of constraints of the form imposed on entries of SSYT of some particular ribbon shape involving two rows, and each residue letter gives rise to a ribbon shape with only one row, so Lemma 3.1.1 applies. Recall that $S(m, n) = h_m h_n - h_{m+n}$, so this gives the expression in terms of complete symmetric functions. □

3.3 k-shuffle posets

The decomposition for k-shuffle posets is quite similar to that of shuffle posets of multisets, but interface pairs are replaced by what we call descent blocks and in k-shuffle posets the ribbon shapes may involve as many as k rows. Recall how each shuffled word $w = w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ gives rise to a product of chains sublattice P_w. We will again partially order these using an interval in the weak order. Let $l(w_i)$ be the **length** of the composition $\alpha^{(i)}$ which records the type of the word w_i. We use the interval in the weak order from the permutation $(l(w_1), \ldots, 1, l(w_1) + l(w_2), \ldots, l(w_1) + 1, \ldots, l(w_1) + \cdots + l(w_k), \ldots, l(w_1) + \cdots + l(w_{k-1}) + 1)$ to the reverse permutation $(l(w_1) + \cdots + l(w_k), \ldots, 1)$; as before, we have covering relations from swapping adjacent positions rather than values. The point is to preserve the order of the letters belonging to each word w_i. For example, if $w_1 = 112$, $w_2 = aaa$ and $w_3 = A$, the

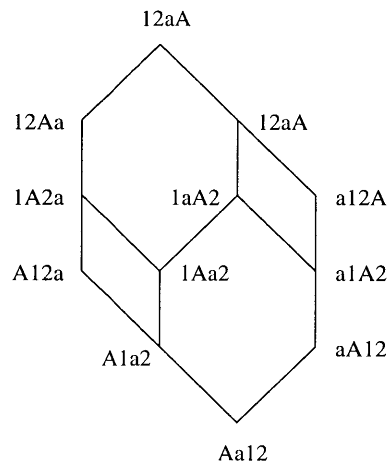

Figure 3-6: The partial order on sublattices indexed by shuffled words

weak order interval we use is given in Figure 3-6

Mimicking the case $k = 2$, each poset chain is assigned to P_w for the earliest w which is consistent with the chain. Proposition 2.2.2 ensures that every chain belongs to some P_w. We also need to make sure that this choice is well-defined of which P_w containing a chain is earliest. This follows from a generalized notion of the interface of a chain which we will call the set of descent blocks of the chain. First recall the notation $w(a) = i$ for $a \in w_i$.

Definition 3.3.1 *A* **descent block** *is a maximal string $u_1 \dots u_j$ of consecutive letters (ignoring repetition of identical letters) in a shuffled word with the property that $w(u_i) > w(u_{i+1})$ for $1 \le i < j$.*

For example, if we replace a word $u = u_1 \dots u_n$ by $w(u_1) \cdots w(u_n)$ to obtain 3114214241, then u has descent blocks represented by $31, 1, 421, 42$ and 41. Let $m_i(\mathbf{b})$ be the multiplicity with which the ith distinct letter in a descent block $\mathbf{b}$ occurs, let $l(\mathbf{b})$ be the number of distinct letters in $\mathbf{b}$ and let $S(m_1(\mathbf{b}), \dots, m_j(\mathbf{b}))$ denote the skew-Schur function of ribbon shape with $m_i(\mathbf{b})$ boxes in row $l(\mathbf{b}) - i + 1$. We claim that the collection of multichains assigned to a product of chains with B as its set of descent blocks contributes

$$\prod_{\mathbf{b} \in B} S(m_1(\mathbf{b}), \dots, m_{l(b)}(\mathbf{b}))$$

to F_P. For example, $S(3,1,2) = s$ would account for a descent block CBA in which $w(C) > w(B) > w(A), m_1(\mathbf{b}) = 3, m_2(\mathbf{b}) = 1$ and $m_3(\mathbf{b}) = 2$.

If B is the set of descent blocks for some P_w then the multichains within P_w which are assigned to it are the multichains which actually involve all the descents in the descent blocks of B. This is exactly the collection of multichains in P_w determined by the system of weak and strong inequalities specifying a product of skew-Schur functions of ribbon shape, each of which has $l(\mathbf{b})$ rows, so Lemma 3.1.1 applies. A descent block consisting of letters $a_1, \ldots, a_l$ with multiplicities $m_1, \ldots, m_l$ will contribute to F_P the skew-Schur function $S(m_1, \ldots, m_l)$, and the contribution of the descent blocks for some P_w are multiplied. Simply note that we have weak inequalities on the order in which identical letters are del-serted and strict inequalities requiring that the last copy of a_i must be del-serted strictly after the first copy of a_{i+1} is del-serted. Otherwise, not all of the necessary descents would occur in the multichain.

Example 3.3.1 *If $w_1 = aabbbcddddde, w_2 = ABBCDDD, w_3 = 1112333344$, then the chains associated to the shuffled word $aa111bbbcA2BBdddddCe333344DDD$ contribute the product of skew-Schur functions $s\ s\ s\ s\ s\ s\ s\ s$ to F_P. The descent blocks are $a, 1b, c, A, 2Bd, Ce, 3$ and $4D$.*

Theorem 3.3.1 *Let $Shuf(w_1, \ldots, w_k)$ be the collection of shuffled words $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ and let $B(w)$ be the collection of descent blocks in a particular $w \in Shuf(w_1, \ldots, w_k)$. Then*

$$F_P = \sum_{w \in Shuf(w_1, \ldots, w_k)} \prod_{\mathbf{b} \in B(w)} S(m_1(\mathbf{b}), \ldots, m_{l(b)}(\mathbf{b})).$$

PROOF. This is an immediate consequence of Lemma 3.1.1, in light of of the above discussion. $\square$

Corollary 3.3.1 *The flag f-vector formula F_P for k-shuffle posets is Schur-positive.*

PROOF. The proof is identical to the $k = 2$ case, namely Corollary 3.2.1. $\square$

Let us mention how to express F_P in terms of complete symmetric functions because Theorem 4.0.3 will relate the characteristic polynomial, zeta polynomial and rank generating function of any finite, ranked poset to its complete homogeneous symmetric function expression. Note that

$$\begin{aligned} S(m_1,\ldots,m_j) = h_{m_1}h_{m_2}\ldots h_{m_j} &- \sum_{i=1}^{j-1} h_{m_1}\ldots h_{m_{i-1}}h_{m_i+m_{i+1}}h_{m_{i+2}}\ldots h_{m_j} \\ &+ \sum_{i=1}^{j-2} h_{m_1}\ldots h_{m_{i-1}}h_{m_i+m_{i+1}+m_{i+2}}h_{m_{i+3}}\ldots h_{m_j} \end{aligned}$$

by the combinatorial definition of the skew-Schur function $s_{\lambda/\mu}$. The term $h_{m_1}\ldots h_{m_j}$ comes from filling each row in a skew-tableau of ribbon shape with a weakly increasing sequence, but then the first sum must be subtracted to account for the fact that a box vertically above another cannot have a (weakly) larger entry, in which case the two rows concatenated would form a weakly increasing sequence; finally, the second sum accounts for overlap in these subtracted terms when three consecutive rows in a ribbon shape are filled with a weakly increasing sequence when the rows are concatenated.

3.4 Noncrossing partition lattices for the classical reflection groups

In the lattice of noncrossing partitions for each of the classical reflection groups, we partially order boolean sublattices comprising the cycles in a homology basis. The decomposition in type A is closely related to a chain-labelling of Stanley [St5, p.7-10] from which he obtains a formula for F_P. We present our chain decomposition point of view for type A before applying it also to the type B,D and interpolating BD noncrossing partition lattices. Chapter 5 will discuss a chain-labelling for the noncrossing partition lattices for the other types, but this may be viewed as a specialization to maximal chains of the decomposition given here.

If we list the numbers $1,\ldots,n$ sequentially on a number line, we may construct a planar tree by drawing a collection of $n-1$ noncrossing arcs above the number line so that each number except 1 is the right endpoint of some arc. The parent of each number is the number to its left connected to it by an arc. By choosing a subset of the arcs in any particular such tree, as in Figure 3-7, we specify a noncrossing partition on $1, \ldots, n$ where two numbers belong to the same component if they are in the same connected component of the resulting graph. The noncrossing partitions given by choosing a subset of the arcs of any particular tree thereby form a boolean sublattice of the noncrossing partition lattice.

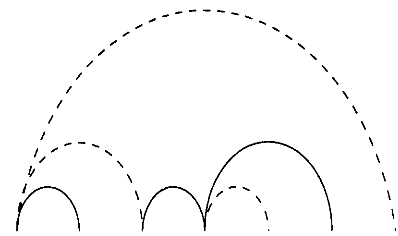

Figure 3-7: A boolean sublattice in NC_n^A

Observe that these boolean sublattices for type A which we depict by trees are implicit to the following EL-labelling for the noncrossing partition lattice.

Theorem 3.4.1 (Björner-Gessel) *If $u \prec v$ then there exist two distinct blocks B_1 and B_2 in the partition u which are merged in v. Let $\lambda(u,v) = max(\min B_1, \min B_2)$. The labelling λ is an EL-labelling for the noncrossing partition lattice.*

Gessel defined this labelling for the partition lattice, and Björner noticed in [Bj2, p. 165] that it could be restricted to the noncrossing partitions. The decreasing chains given by this EL-labelling determine boolean sublattices which coincide with the boolean sublattices which we have specified by trees. To see this, note that the right endpoints of our trees are the edge-labels of the EL-labelling.

In order to give a chain decomposition, we partially order these trees. Let the tree consisting of arcs $i, i+1$ for $1 \leq i \leq n-1$ be the minimal element in this partial order. If two trees u and v agree except that u has arcs i, j and j, k while v has arcs i, j and i, k for some $i < j < k$, then we let $u \prec v$.

Our chain decomposition comes from assigning each chain to the earliest of these sublattices which contains it. The connection to the EL-labelling in Theorem 3.4.1

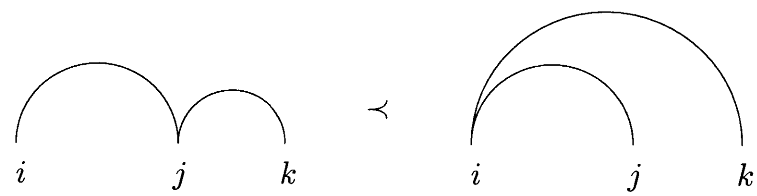

Figure 3-8: Covering relations for trees in NC_n^A

guarantees that every poset chain belongs to at least one of these boolean sublattices, and we claim that the notion of "earliest" boolean sublattice containing a particular chain is well-defined; if there are two incomparable trees such that a chain may be constructed using only the arcs in either one, then there is some tree earlier than both from which the chain may also be constructed.

For each tree, note that the multichains assigned to the corresponding boolean sublattice satisfy the following property: if two arcs have the same left endpoint, then the arc with right endpoint farther to the right must be inserted strictly earlier than the other arc. Otherwise, there would be an earlier tree which has these arcs ij and ik replaced by arcs ij and jk, as in Figure 3-8. If we let $m_i(t)$ be the number of arcs with i as their left endpoint in our tree t, then we are restricting to chains in a boolean lattice satisfying the strict inequalities

$$\begin{aligned} a_{1,1} &< \cdots < a_{1,m_1(t)} \\ a_{2,1} &< \cdots < a_{2,m_2(t)} \\ &\cdots \\ a_{n,1} &< \cdots < a_{n,m_n(t)}. \end{aligned}$$

The variables represent arcs to be inserted with the convention that $a_{i,j}$ has i as its left endpoint and is the jth such arc when these are ordered top to bottom (i.e. so that the arc with the rightmost right endpoint comes first). We apply Lemma 3.1.1 to recover the following formula, given in [St5, p.6] and originally determined in [Ed, p.173-174]. If P is the noncrossing partition lattice NC_{n+1}, then

$$F_P = \sum_{t \in T} e_{m_1(t)} e_{m_2(t)} \cdots e_{m_n(t)}$$

where T is the collection of rooted unlabelled planar trees.

Let us observe how the trees we discuss are related to the vertex-labelled trees of [ES, p.9-13]. For each saturated chain assigned to a particular tree by our decomposition, we may label the arcs with step at which they are inserted in traversing the saturated chain from $\hat{0}$ to $\hat{1}$. Thus, each vertex is labelled with an ordered pair, namely the label already associated to the vertex together with the edge label between the vertex and its parent in the tree.

Next we generalize this decomposition to the noncrossing partition lattices for the other classical reflection groups. One may define circular tree diagrams for type B which again give rise to boolean sublattices. Figure 3-9 provides an example. Circular tree diagrams are required to preserve 180 degree rotational symmetry. If there is an arc i, j with i no more than 180 degrees counterclockwise from j, then we consider i to be the parent of j in this treelike structure. Every node must have a parent, and we denote an arc by i, j where i is the parent of j. Circular tree diagrams are partially ordered in a similar fashion to the type A situation: we say $u \prec v$ if u and v agree except that u has arcs i, j and j, k while v has arcs i, j and i, k.

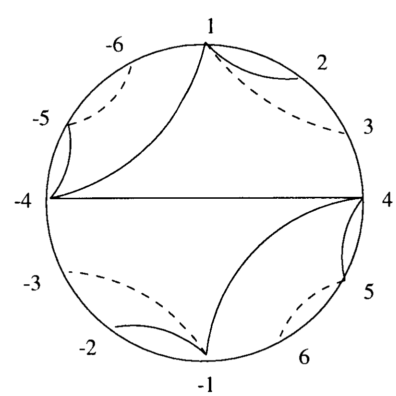

Figure 3-9: A boolean sublattice in NC_n^B

We get a system of strict inequalities, just as in the type A case. Summing F_P over pieces of a chain decomposition yields the flag f-vector formula of Corollary 3.4.1. This agrees with a flag f-vector formula in [Re, p.13]. In the next two corollaries, we sum over compositions of n, denoted by α, where α_i will count the number of digits which are equal to i.

Corollary 3.4.1 *If P is the type B noncrossing partition lattice $NC^B(n)$ and T is the collection of possible tree diagrams, then*

$$F_P = \sum_{t \in T} e_{m_1(t)} \ldots e_{m_n(t)} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha_1 + \cdots + \alpha_n = n}} e_\alpha.$$

For example, if $P = NC^B(3)$ then $F_P = e_1^3 + 6e_1e_2 + 3e_3$. In the case of interpolating BD noncrossing partitions, the allowable circular tree diagrams are those which do not include an arc from i to $-i$ for any $i \in S$. The chain decomposition for type B restricts to this case because the partial order on circular tree diagrams does not include any covering relations $u \prec v$ where u is forbidden and v is a legal circular tree diagram.

Corollary 3.4.2 *If P is the interpolating BD noncrossing partition lattice NC_S^{BD}, then*

$$F_P = \sum_{t \in T} e_{m_1(t)} \ldots e_{m_n(t)} = \sum_{\substack{\alpha \in \mathbb{N}^n \cap PF_S \\ \alpha_1 + \cdots + \alpha_n = n}} e_\alpha$$

for $PF_S = \bigcap_{i \in S} \{\alpha | \alpha_i + \cdots + \alpha_j < j - i + 1$ *for some* $i \leq j \leq n$ *or* $\alpha_i + \cdots + \alpha_n + \alpha_1 + \cdots + \alpha_j < j + n - i + 1$ *for some* $j < i\}$.

3.5 Graded monoid posets

Recall from Chapter 2 how a graded monoid poset is obtained from the ring $k[\Lambda] \cong k[x_1, \ldots, x_n]/I_\Lambda$ where I_Λ is the toric ideal of syzygies. For any fixed poset interval $[\hat{0}, m]$, we will give a chain decomposition based on a Gröbner basis for I_Λ. This Gröbner basis gives us a way to partially order sublattices each of which is a product of chains. For any monomial m in the image of the map $\phi : k[x_1, \ldots, x_n] \to k[z_1, \ldots, z_d]$, we have a graded monoid poset interval from $\hat{0}$ to m, and the product of chains sublattices are given by the distinct monomials $f \in k[x_1, \ldots, x_n]$ such that $\phi(f) = m$.

Peeva, Reiner and Sturmfels use noncommutative Gröbner bases in [PRS, p.6-8] to give a shelling of the order complex of $[\hat{0}, m]$ based on a quadratic Gröbner basis for a related ideal J_Λ in noncommuting variables, when such a quadratic Gröbner basis exists. Each saturated chain in a graded monoid poset naturally corresponds to a monomial in the ring $k\langle y_1, \ldots, y_n\rangle$ in noncommuting variables. Let $\psi : k\langle y_1, \ldots, y_n\rangle \to k[x_1, \ldots, x_n]$ send y_i to x_i. The ideal J_Λ in $k\langle y_1, \ldots, y_n\rangle$ is defined by $J_\Lambda = \psi^{-1}(I_\Lambda)$. The elements of any particular fiber of ψ will all belong to the same piece of our chain decomposition, so commutative Gröbner bases for I_Λ will suffice for our purposes.

Let us choose some fixed Gröbner basis B for the toric ideal $I_\Lambda = ker(\phi)$. We first consider the case where the elements of $\phi^{-1}(m)$ are all squarefree. For each $f \in \phi^{-1}(m)$, let $L_{A(f,B)}$ be the intersection lattice of the subspace arrangement given by the subspaces $a_1 = a_2 = \ldots = a_k$ for each monomial $a_1 \ldots a_k$ which divides f and also is the leading term of some element of B.

Theorem 3.5.1 *If P is a graded monoid poset interval $[\hat{0}, m]$ with $m \in im(\phi)$ and the factorizations $f \in \phi^{-1}(m)$ for the maximal element m are squarefree then*

$$F_P = \sum_{f \in \phi^{-1}(m)} \sum_{u \in L_{A(f,B)}} \mu(\hat{0}, u) p_{type(u)}$$

where μ is the Möbius function on the intersection lattice $L_{A(f,B)}$ which is defined above.

PROOF. This follows from a straightforward inclusion-exclusion argument. Note that each factorization $f \in \phi^{-1}(m)$ gives rise to a boolean sublattice which contributes a power-sum symmetric function p_{1^n} to F_P since $p_{1^n} = h_{1^n}$. Möbius functions will account for overlap. A factorization $f \in \phi^{-1}(m)$ is ordered earlier than another factorization $f' \in \phi^{-1}(m)$ if f may be obtained from f' by a sequence of reduction steps $f' = f_1 \to f_2 \to \cdots \to f_k = f$ where each step $f_i \to f_{i+1}$ consists of replacing a Gröbner basis leading term which divides f_i by an equivalent smaller order term to yield f_{i+1}. Each poset chain is assigned to the earliest boolean sublattice containing it where the boolean sublattices are ordered by this order on the factorizations specifying them. By nature of a Gröbner basis, the notion of which boolean sublattice containing a chain comes earliest is well-defined; for each jump in a chain, we choose the monomial in $k[x_1, \ldots, x_n]$ which represents some saturated chain in the interval and is reduced as much as possible.

Subspace arrangements naturally describe how each boolean sublattice intersects with the union of earlier ones. Let S be the sphere in the order complex given by a particular boolean sublattice and let f be the corresponding monomial in $\phi^{-1}(m)$. The overlap of S with the union of the earlier spheres in the order complex is the restriction to S of a subspace arrangement which is a subarrangement of the type A Coxeter hyperplane arrangement, as observed in [HRW, p.5-6]. The subspaces defining this arrangement are given by the leading terms in B which divide f as follows. If $a_1 \ldots a_k$ is such a leading term, then the intersection of S with the union of earlier spheres includes the subspace $a_1 = \cdots = a_k$. The poset chains within S belonging to any of these subspaces are exactly the ones which may be reduced to earlier factorizations, and therefore have already been counted in our formula for F_P. As discussed at the beginning of the chapter, the correspondence between chains and subspaces comes from letting $a_i = a_j$ whenever both occur in a single jump in a chain. When a Gröbner basis leading term divides the monomial corresponding to such a jump, then it is possible to apply a relation to obtain a smaller order equivalent factorization for the jump, implying the chain would have already been counted earlier in the decomposition.

To show that $\sum_{u \in L_{A(f,B)}} \mu(\hat{0}, u) p_{type(u)}$ accounts for exactly the new chains in the boolean sublattice given by f, recall that the intersection of the subspace of type λ with a sphere contributes p_λ to F_P. Furthermore, multiplying by the Möbius functions of $L_{A(f,B)}$ ensures that each old chain is counted with multiplicity zero and each new chain with multiplicity one. This is because $p_{type(u)}$ counts chains with jumps given by u or with any of these jumps merged, so the chains in a subspace v are counted in $p_{type(u)}$ for each $u \leq v$. Hence, the chains in each subspace $v \neq \hat{0}$ are counted with multiplicity $\sum_{0 \leq u \leq v} \mu(\hat{0}, u) = 0$ while the new chains are only counted in the term $\mu(\hat{0}, \hat{0}) p_{1^n}$ which means they are each counted exactly once. $\square$

Example 3.5.1 *Let $a = uv, b = vx, c = yz, d = uy, e = xy$ and $f = vz$. Consider the poset interval from 1 to $uv^2xyz = abc = bdf = aef$, so I_Λ is generated by the Gröbner basis $\underline{df} - ac, \underline{bd} - ae, \underline{ef} - bc$ with leading terms underlined. For $[\hat{0}, uv^2xyz]$,*

$$F_P = p_1^3 + (p_1^3 - p_1p_2) + (p_1^3 - p_1p_2 - p_1p_2 + p_3).$$

Each factorization contributes one copy of p_1^3. A copy of p_1p_2 is subtracted for the intersection of the aef sphere with the earlier abc sphere in the $e = f$ hyperplane. Two copies of p_1p_2 are subtracted to account for the $b = d$ and $d = f$ hyperplanes in the bdf sphere, but these hyperplanes share the subspace $b = d = f$, so p_3 must be added back.

We next address the case where not all the $f \in \phi^{-1}(m)$ are necessarily squarefree; this will involve a variant of the Möbius function which we denote by μ' and which is studied in more detail in Chapter 6. If P is the poset of partitions of the multiset $\{a_1^{n_1}, \dots, a_k^{n_k}\}$, reverse ordered by refinement, then we define $\mu'(\hat{0}, v)$ as follows. If v is a single block B_i, then let $\mu'(\hat{0}, v) = -\sum_{\hat{0} \leq u < B_i} \mu'(\hat{0}, u)$. When v is a partition into multiple blocks $B_1, \dots, B_k$, then let

$$\mu'(\hat{0}, B_1 | B_2 | \dots | B_k) = \prod_{1 \leq i \leq k} \mu'(\hat{0}, B_i).$$

In Chapter 6, we will prove that

$$\mu'(\hat{0}, ab_1^{n_1} \dots b_k^{n_k}) = \frac{(n-1)!}{\prod_{i=1}^{k} n_i!}.$$

In addition, we will show that $\mu'(\hat{0}, a^n) = 0$ unless n is a power of 2, in which case $\mu'(\hat{0}, a^n) = (-1)^{n-1}$. For each multiset partition u, we interpret $\mu'(\hat{0}, u)$ as the Euler characteristic of a shellable cell complex closely related to the order complex of the multiset partition poset given by u. Hence, $\mu'(\hat{0}, u)$ counts spheres of top dimension in a homology basis for this complex.

Question 3.5.1 *Is there an explicit formula for* $\mu'(\hat{0}, a_1^{n_1} \dots a_k^{n_k})$ *in general?*

To give a flag f-vector formula for all graded monoid posets, we will use the funcion μ' in a sieve argument similar to the one in the squarefree case, but this will require the following lemma. If v is a refinement of u, then let $Sub(u,v)$ count the number of ways of choosing which types of blocks in u belong to each component of a less refined multiset partition v. For example, if $u = a|a|aa$ and $v = aa|aa$, then $Sub(u,v) = 2$, but if v is replaced by $aaaa$ then $Sub(u,v) = 1$.

Lemma 3.5.1 *If* $v \neq \hat{0}$, *then*

$$\sum_{\hat{0} \leq u \leq v} \mu'(\hat{0}, u) Sub(u, v) = 0.$$

Proof. We may assume v has multiple blocks because for a single block B we have $\mu'(\hat{0}, B) = -\sum_{\hat{0} \leq u < B} \mu'(\hat{0}, u)$, and in this case $Sub(u, B) = 1$ for each $\hat{0} \leq u < B$. Let us assume v has blocks $B_1, \dots, B_k$ for $k > 1$.

Note that

$$\sum_{\emptyset \subsetneq S \subsetneq \{1,\dots,k\}} (-1)^{|S|-1} \prod_{i \in S} \mu'(\hat{0}, B_i) \prod_{i \notin S} \left(\sum_{\hat{0} \leq u \leq B_i} \mu'(\hat{0}, u) \right) = 0,$$

because $\sum_{\hat{0} \leq u \leq B_i} \mu'(\hat{0}, u) = 0$ for each B_i. We may therefore add this to the next expression without affecting its value.

If v has blocks $B_1, \ldots, B_k$ for $k > 1$, then

$$\begin{aligned}
\mu'(\hat{0}, v) &= \mu'(\hat{0}, B_1|\ldots|B_k) \\
&= \prod_{i=1}^{k} \mu'(\hat{0}, B_i) \\
&= \prod_{i=1}^{k} \left(- \sum_{\hat{0} \leq u < B_i} \mu'(\hat{0}, u) \right) \\
&= (-1)^k \prod_{i=1}^{k} \left(\sum_{\hat{0} \leq u < B_i} \mu'(\hat{0}, u) \right) \\
&\quad + \ (-1)^k \sum_{\emptyset \subsetneq S \subsetneq \{1,\ldots,k\}} (-1)^{|S|-1} \prod_{i \in S} \mu'(\hat{0}, B_i) \prod_{i \notin S} \left(\sum_{\hat{0} \leq u \leq B_i} \mu'(\hat{0}, u) \right) \\
&= (-1)^k \sum_{\hat{0} \leq u < v} \mu'(\hat{0}, u) Sub(u, v) \\
&\quad + \ (-1)^k \mu'(\hat{0}, B_1|\ldots|B_k) \left(\sum_{i=1}^{k-1} (-1)^{i-1} \binom{k}{i} \right)
\end{aligned}$$

If k is even, then we have $\sum_{i=1}^{k-1}(-1)^{i-1}\binom{k}{i} = 2$, which implies

$$\mu'(\hat{0}, v) = (-1)^k \sum_{\hat{0} \leq u < v} \mu'(\hat{0}, u) Sub(u, v) + 2\mu'(\hat{0}, B_1|\ldots|B_k).$$

Subtracting $\mu'(\hat{0}, v)$ from both sides yields the result, since $(-1)^k = 1$ for k even.

For k odd, $\sum_{i=1}^{k-1}(-1)^{i-1}\binom{k}{i} = 0$ and $(-1)^k = -1$, so we have

$$\mu'(\hat{0}, v) = - \sum_{\hat{0} \leq u < v} \mu'(\hat{0}, u) Sub(u, v),$$

as desired. □

Our use of symmetric functions to account for many chains at once will introduce factors of $Sub(u, v)$ into our sieve argument. In the non-square-free case, the formula for F_P will not only involve the generalized Möbius functions μ' on multiset partition posets, but will also rely on a sort of multiset analogue of intersection lattices of subspace arrangements. Let us define these posets and their generalized Möbius functions next.

Given a collection C of leading terms in a Gröbner basis which are all squarefree and a monomial $a_1 \cdots a_n$, this specifies a subspace arrangement given by the subspaces $a_{i_1} = \cdots = a_{i_k}$ for each leading term $a_{i_1} \cdots a_{i_k}$ which divides $a_1 \cdots a_n$. We generalize this to the nonsquarefree case where there is a collection C' of leading terms each of which divides some $a_1^{m_1} \dots a_l^{m_l}$.

Definition 3.5.1 *The minimal element in a* **multiset intersection poset** *is the partition of a multiset* $\{a_1^{m_1}, \dots, a_k^{m_k}\}$ *into trivial blocks while the single block* $a_1^{m_1} \cdots a_l^{m_l}$ *is the maximal element. For each Gröbner basis leading term* $a_{i_1}^{n_{i_1}} \cdots a_{i_j}^{n_{i_j}}$ *which divides* $a_1^{m_1} \cdots a_l^{m_l}$*, there is a poset element* u *which covers* $\hat{0}$ *and which has one nontrivial block* $a_{i_1}^{n_{i_1}} \cdots a_{i_j}^{n_{i_j}}$*. A* **legal merge step** *consists of merging some collection* $B_1, \dots, B_s$ *of blocks in* u *where there is some Gröbner basis leading term which may be split into a product of nontrivial monomials* $m_1, \dots, m_s$ *such that* $m_r \subseteq B_r$ *for* $1 \le r \le s$*. If* v *may be obtained from* u *by a sequence of legal merge steps, then we let* $u \le v$*.*

These multiset intersection posets are not in general lattices, so in particular they are not intersection lattices. However, legal merge steps are intended to play the role of intersection of subspaces. We define $\mu'_P(\hat{0}, B_1|\dots|B_k)$ to be $\prod_{i=1}^k \mu'_{P_i}(\hat{0}, B_i)$ with covering relations in P_i coming only from legal merge steps in the interval $[\hat{0}, B_i]$. Let $\mu'(\hat{0}, B) = -\sum_{\hat{0} \le u < B} \mu'(\hat{0}, u)$ for each block B.

Lemma 3.5.1 generalizes directly to arbitrary multiset intersection posets by restricting throughout the proof to elements and covering relations belonging to a multiset intersection poset. The proof of Theorem 3.5.2 is similar to the square-free case, but with power-sum symmetric functions replaced by complete symmetric functions in variables raised to powers and using the fact that $\sum_{\hat{0} \le u \le v} \mu'(\hat{0}, u) Sub(u, v) = 0$ rather than $\sum_{\hat{0} \le u \le v} \mu(\hat{0}, u) = 0$. Note that $h_1(x_1^n, x_2^n, \dots) = p_n$ and $\mu' = \mu$ in the squarefree case, so the formula below specializes to the squarefree case. We choose to express $h_i(x_1^j, x_2^j, \dots)$ in terms of plethystic substitution. We let $MP(f, B)$ denote the multiset intersection poset given by maximal monomial f and Gröbner basis

B. Let $T(u)$ be the set of distinct blocks occuring in a multiset partition u and let $mult(t)$ be the multiplicity with which any particular block occurs in a multiset partition. Let us make a change of variables just within Theorem 3.5.2 by replacing the ring $k[x_1, \dots, x_n]$ by the ring $k[a_1, \dots, a_n]$, so as to avoid confusion with the variables arising in symmetric functions that account for chains; we treat $k[a_1, \dots, a_n]$ as the preimage of the map ϕ in the obvious way.

Theorem 3.5.2 *If P is the graded monoid poset interval $[\hat{0}, m]$, then*

$$F_P = \sum_{f \in \phi^{-1}(m)} \sum_{u \in MP(f,B)} \mu'(\hat{0}, u) \prod_{t \in T(u)} p_{|t|} \circ h_{mult(t)}(x_1, x_2, \dots).$$

PROOF. The combinatorial decomposition is just as in the squarefree case, but with product of chains sublattices replacing the boolean sublattices of the square-free case. Each factorization $a_{i_1}^{\lambda_1} \dots a_{i_k}^{\lambda_k}$ with $i_1 < \dots < i_k$ gives rise to a product of chains sublattice $C_{\lambda_1+1} \times \dots \times C_{\lambda_k+1}$ which contributes h_λ to F_P. To account for overlap, these products of chains are partially ordered as in the square-free case, using the Gröbner basis B on I_Λ. The symmetric function

$$\prod_{t \in T(u)} h_{mult(t)}(x_1^{|t|}, x_2^{|t|}, \dots)$$

accounts for all chains with jumps determined by the multiset partition u or with collections of these jumps merged into single jumps. The exponent $|t|$ keeps track of the size of jumps while the subscript $mult(t)$ allows for repetition of identical jumps. We choose to write $h_n(x_1^r, x_2^r, \dots)$ as $h_n \circ p_r(x_1, x_2, \dots)$, using plethystic substitution (cf. [Ma, I,8 ex. 7]). Since F_P is multiplicative, we take a product over the different types of blocks occuring in the multiset partition u. The coefficients $\mu'(\hat{0}, u)$ are chosen so that each new chain is included once and each chain that was already counted in an earlier product of chains is counted with multiplicity 0; the desired coefficients result from a sieve argument based on Lemma 3.5.1. □

For example, if $f = a^4$ and the only leading term in B which divides f is a^2, then the contribution to F_P from the product of chains C_5 specified by f is

$$p_1 \circ h_4 - (p_2 \circ h_1)(p_1 \circ h_2) + p_2 \circ h_2 - p_4 \circ h_1,$$

or equivalently,

$$h_4(x_1, x_2, \dots) - h_1(x_1^2, x_2^2, \dots) h_2(x_1, x_2, \dots) + h_2(x_1^2, x_2^2, \dots) - h_1(x_1^4, x_2^4, \dots).$$

The terms come from the jump collections $\{a, a, a, a\}$, $\{a, a, a^2\}$, $\{a^2, a^2\}$ and $\{a^4\}$, respectively.

Question 3.5.2 *Are there better formulas for F_P for graded monoid posets in special cases? Is there some monomial term order which gives a particularly nice formula, at least in special cases?*

Chapter 4

Applications to poset combinatorics, structure and topology

In this chapter, we give applications of the chain decompositions provided in the previous chapter. We focus on shuffle posets of multisets and k-shuffle posets, but we believe a similar analysis of noncrossing partitions for the classical reflection groups should be possible. However, many of the results would likely duplicate work of Edelman, Simion, Reiner, et al. We do provide symmetric chain decompositions for the noncrossing partition lattices for the classical reflection groups and a proof of supersolvability for noncrossing partitions of type A. In these cases and in shuffle posets, the symmetric chain decompositions come from specializing chain decompositions to 1-chains; proofs of supersolvability involve finding a saturated chain that lies in each of the product of chains subposets specifying our chain decomposition and then showing that this is an M-chain. Let us give two general results before turning specifically to shuffle posets.

When F_P is a symmetric function, then the coefficient of h_{1^n} in its complete homogeneous symmetric function expression is the Möbius function $\mu_P(\hat{0}, \hat{1})$ multiplied by a sign. This ties in with our topological chain decompositions into overlapping boolean subposets which each contribute h_{1^n} to F_P before we account for their (contractible) overlap.

Proposition 4.0.1 *If P is a poset of rank n with flag f-vector expression F_P a symmetric function, then the coefficient of h_{1^n} in F_P is the Möbius function $\mu_P(\hat{0},\hat{1})$ multiplied by the sign $(-1)^n$.*

PROOF. Ehrenborg expressed $\mu_P(\hat{0},\hat{1})$ as a linear function of F_P in [Eh, p.17]; let g be this function. Note that $g((h_1)^{n+m}) = (-1)^{m+n}$ and $g(h_\lambda) = 0$ for all other λ because $g(h_\lambda) = \mu_P(\hat{0},\hat{1})$ for P the product of chains C_λ. Let $F_P = \sum_{\lambda\vdash n} c_\lambda h_\lambda$. Then $\mu_P(\hat{0},\hat{1}) = g(F_P) = g(\sum_{\lambda\vdash n} c_\lambda h_\lambda) = \sum_{\lambda\vdash n} c_\lambda g(h_\lambda) = (-1)^n c_{1^n}$. □

More generally, one may obtain from the complete homogeneous symmetric function expression for F_P several combinatorial formulas by simple substitutions. We note that Ehrenborg expressed the zeta polynomial and characteristic polynomial more formally as Hopf algebra homomorphisms in [Eh, p.16-17]. Let $h_\lambda = h_{\lambda_1}\dots h_{\lambda_k}$, let M_P be the number of maximal chains in a poset P, let Ω_P be the number of poset elements, let $\Omega_P(q)$ be the rank generating function, let $Z_P(s)$ be the zeta polynomial, counting multichains $\hat{0} \le x_1 \le \cdots \le x_s = \hat{1}$ and let $\chi_P(t)$ be the characteristic polynomial $\sum_{u\in P} \mu(\hat{0},u)t^{n-\mathrm{rk(u)}}$ In the expression for $\Omega_P(q)$, let $[n]_q = (1-q^n)/(1-q)$, namely the q-analogue of n.

Theorem 4.0.3 *If $F_P = \sum_{\lambda\vdash n} c_\lambda h_\lambda$, then*

$$\begin{aligned}
\Omega_P &= \sum_{\lambda\vdash n} c_\lambda(\lambda_1+1)\dots(\lambda_k+1)\\
\Omega_P(q) &= \sum_{\lambda\vdash n} c_\lambda[\lambda_1+1]_q\dots[\lambda_k+1]_q\\
M_P &= (\mathrm{rk(P)!})\sum_{\lambda\vdash \mathrm{n}} \mathrm{c}_\lambda\frac{1}{\lambda_1!}\cdots\frac{1}{\lambda_\mathrm{k}!}\\
Z_P(s) &= \sum_{\lambda\vdash n} c_\lambda\binom{\lambda_1+s-1}{s-1}\cdots\binom{\lambda_k+s-1}{s-1}\\
\chi_P(t) &= t^{\mathrm{rk(P)}}\sum_{\lambda\vdash n} c_\lambda\left(\frac{1}{1-t}\right)^k
\end{aligned}$$

PROOF. Linearity allows us to substitute $d+1$ for h_d to obtain Ω_P. This is because a chain of rank d satisfies $F_P = h_d$ and $\Omega_P = d+1$, and also because both F_P and Ω_P are multiplicative functions. Similarly, we substitute $[d+1]_q$ and $\binom{d+s-1}{s-1}$ for $d+1$ to obtain the rank generating function and zeta polynomial, respectively. In the formula counting maximal chains, M_P is not multiplicative, so $M_P/(\mathrm{rk(P)})!$ is used instead, introducing the factor of $(\mathrm{rk(P)})!$. Likewise, to obtain the characteristic polynomial, notice that $\sum_{u\in P}\mu(\hat{0},u)(1/t)^{\mathrm{rk(u)}}$ is a multiplicative function equalling $\chi_P(t)/t^{\mathrm{rk(P)}}$, so an extra factor of $t^{\mathrm{rk(P)}}$ is introduced. □

4.1 Shuffle posets of multisets

Denote by (1) the formula for F_P for shuffle posets of multisets

$$\sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j}(h_{\alpha_{a_i}}h_{\beta_{b_i}} - h_{\alpha_{a_i}+\beta_{b_i}})\right)\left(\prod_{\substack{i\notin\{a_1,\ldots,a_j\}\\ 1\le i\le l(\alpha)}} h_{\alpha_i}\right)\left(\prod_{\substack{i\notin\{b_1,\ldots,b_j\}\\ 1\le i\le l(\beta)}} h_{\beta_i}\right).$$

Recall from [Gr, p.200] that a chain is w_1-*terminal* (w_2-*terminal*) if each chain element involving the last letter of w_1 (w_2) has this letter occuring last, i.e. after all letters of w_2 (w_1) appearing in the shuffled word. This leads to the following recursive formula.

Lemma 4.1.1 *The shuffle posets of multisets satisfy the recurrence*

$$F_{\alpha,\beta} = F_{\alpha-\alpha_k,\beta}h_{\alpha_k} + F_{\alpha,\beta-\beta_l}h_{\beta_l} - F_{\alpha-\alpha_k,\beta-\beta_l}h_{\alpha_k+\beta_l} \tag{2}$$

for $\alpha = (\alpha_1,\ldots,\alpha_k)$ *and* $\beta = (\beta_1,\ldots,\beta_l)$.

PROOF. Decompose the space of chains into those which are w_1-terminal and those which are w_2-terminal, subtracting for overlap from chains which never involve the last letters of w_1 and w_2 simultaneously and which are both w_1-terminal and w_2-terminal. Note that restricting to chains in which a letter always appears last yields the product of a chain of rank equalling the multiplicity with which that letter appears and a shuffle poset not involving that letter. We apply $F_{P\times Q} = F_P F_Q$ to obtain each of the three terms on the right side of (2), recalling that $F_P = h_r$ when P is a chain of rank r. □

Lemma 4.1.1 yields an easy inductive proof of (1). Another consequence is the following generating function.

Theorem 4.1.1 *Summing over compositions α and β indexed by monomials in non-commutative variables u_i and v_j with relations $u_i v_j = v_j u_i$ for $i, j > 0$ yields*

$$\sum_{\alpha,\beta} F_{\alpha,\beta} u_{\alpha_1} \ldots u_{\alpha_k} v_{\beta_1} \ldots v_{\beta_l} = \frac{1}{1 - \sum_{i>0}(u_i + v_i)h_i + \sum_{i,j>0} u_i v_j h_{i+j}}.$$

Non-commutative variables are used because F_P depends on the order of the parts of the compositions α and β. This agrees with a result of [SS, p.9] when we express h_1 as e_1, h_2 as $e_1e_1 - e_2$ and let $u_1 = u$, $v_1 = v$ and set $u_i = v_i = 0$ for $i > 1$.

The symmetry of the right hand side in Theorem 4.1.1 in the alphabets u and v immediately implies that $F_{\alpha,\beta} = F_{\beta,\alpha}$. This generalizes to k-shuffle posets to yield a result which in that case does not follow from local rank-symmetry (as it does here). Now we turn to structural properties of shuffle posets of multisets.

Proposition 4.1.1 *Shuffle posets of multisets are lattices.*

Proof. Greene's argument in [Gr, p.193] for traditional shuffle posets indicates which letters must occur in what order in $u \wedge v$ and in $u \vee v$ if we temporarily ignore multiplicity of identical letters in u and v. Any letter of w_1 which does not appear in both u and v also will not appear in $u \vee v$. In addition, any letter of w_1 which appears in u and v in an inconsistent fashion, cannot appear in $u \vee v$. All other letters from $u|_{A_1}$ do appear in $u \vee v$ in addition to the letters in $u|_{A_2} \cup v|_{A_2}$. To obtain $u \vee v$, we next must insert any letter from w_2 which appears in v but not in u. There is a well-defined way to do this so that the resulting word is comparable with both u and v, and this is made explicit in [Gr, p.193]. Finally, the multiplicity of letters in $u \vee v|_{A_1}$ (resp. $u \vee v|_{A_2}$) needs to be the minimum (resp. maximum) of their multiplicities in u and in v. The construction of $u \wedge v$ is similar. $\square$

Proposition 4.1.2 *If $\alpha = 1^m$ and $\beta = 1^n$, then $\mu_{W_{\alpha,\beta}}(\hat{0}, \hat{1}) = (-1)^{m+n}\binom{m+n}{m}$. Otherwise, $\mu_{W_{\alpha,\beta}}(\hat{0}, \hat{1}) = 0$.*

PROOF. The case where $\alpha = 1^m, \beta = 1^n$ is proven in various ways in [Gr, p.206], [SS, p.16] and [BlS, p.107]. Proposition 4.0.1 expresses the Möbius function of any flag-symmetric poset as the coefficient of h_1^n in the complete symmetric function expression for F_P, so we need only inspect the formula in Theorem 3.2.1. $\square$

Note that every interval in a shuffle poset of multisets is a product of smaller shuffle posets of multisets. Since the Möbius function is multiplicative, $\mu(u, v)$ may be determined from Proposition 4.1.2 for arbitrary $u \leq v$. A canonical way of associating products of shuffle posets to intervals is discussed in [SS, p.8]. This applies similarly to shuffle posets of multisets.

Proposition 4.1.3 *Shuffle posets of multisets are EL-shellable.*

PROOF. Greene's EL-labelling for traditional shuffle posets given in [Gr, p.205-206] generalizes directly. Label poset edges with the letter of w_1 to be deleted or the letter of w_2 to be inserted. Choose any total order on the alphabets A_1 and A_2 such that all the letters in A_1 come before all the letters in A_2. Just as in traditional shuffle posets, there will be a unique increasing chain on each interval because there is no choice of position to make when deleting before inserting. $\square$

Corollary 4.1.1 *Shuffle posets of multisets are Cohen-Macaulay.*

When $\mu(\hat{0}, \hat{1}) = 0$, shellability implies the collapsibility of the order complex since the reduced homology groups all vanish.

Corollary 4.1.2 *The order complex of $W_{\alpha,\beta}$ is collapsible unless $\alpha = 1^m$ and $\beta = 1^n$ for some $m, n \in \mathbb{N}$.*

PROOF. Shuffle posets of multisets consist of overlapping products of chains given by the shuffled words $w_1 \sqcup\!\sqcup w_2$. When $\alpha = 1^m$ and $\beta = 1^n$ for $m, n \in \mathbb{N}$ then these are boolean and give rise to a homology basis for the order complex. The decreasing chains in Greene's EL-labelling specify these boolean sublattices. Otherwise, the reduced homology groups all vanish since the order complex is Cohen-Macaulay and the Möbius function is 0. This together with shellability implies collapsibility. $\square$

Recall the notion of interface from [Gr, p.195-196], which also came into play in Section 3.2. Greene introduced this to construct a symmetric chain decomposition for posets of shuffles, and the same argument works for shuffle posets of multisets in general. Note that this decomposition may be viewed as a specialization of our chain decomposition to poset elements, considered as 1-chains.

Proposition 4.1.4 *Shuffle posets of multisets have symmetric chain decompositions.*

PROOF. If we specialize the chain decomposition of Chapter 3 to poset elements, the result is a decomposition into symmetrically placed products of chains. Each shuffled word $w_1 \sqcup\!\sqcup w_2$ gives rise to a product of chains subposet, and the 1-chains that get assigned to it are the ones satisfying certain constraints. Namely, if $w_1 \sqcup\!\sqcup w_2$ has interface pairs a_i, b_i for $1 \le i \le j$ and if $m(a_i)$ denotes the multiplicity with which a_i occurs in w_1, then we have the constraints $b_{i_1} < a_{i_{m(a_i)}}$ for $1 \le i \le j$. The 1-chains in the product of chains given by $w_1 \sqcup\!\sqcup w_2$ which satisfy these constraints form a symmetrically placed product of chains from rank j to rank $n - j$ in a poset of rank n. This is equivalent to partitioning poset elements according to interface as in Greene's proof for traditional shuffle posets [Gr, p.195-198]. $\square$

Next we deduce supersolvability from the fact that a single maximal chain belongs to all the product of chains sublattices specifying our chain decomposition.

Proposition 4.1.5 *Shuffle posets of multisets are supersolvable.*

PROOF. We claim that any maximal chain in which all the letters of w_1 are deleted before any letters of w_2 are inserted is an M-chain. Denote such a chain by C, and note that each such C is consistent with every possible shuffled word $w = w_1 \sqcup\!\sqcup w_2$, which means it belongs to each of the product of chains subposets P_w. The claim then follows from the facts that every poset chain belongs to some P_w and that each P_w is modular. $\square$

We conclude this section with a collection of formulas for shuffle posets of multisets which provide an analogue to Theorem 3.4 of Greene in [Gr, p.195].

Theorem 4.1.2 *The following formulas hold for the shuffle poset of multisets* $W_{\alpha,\beta}$ *with* $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_l)$, $m = \sum_{i=1}^{k} \alpha_i$ *and* $n = \sum_{i=1}^{l} \beta_i$.

$$\Omega_{\alpha,\beta} = \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j} \alpha_{a_i}\beta_{b_i}\right)\left(\prod_{i\notin\{a_1,\ldots,a_j\}} (\alpha_i+1)\right)\left(\prod_{i\notin\{b_1,\ldots,b_j\}} (\beta_i+1)\right)$$

$$\Omega_{\alpha,\beta}(q) = \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j}([\alpha_{a_i}+1]_q[\beta_{b_i}+1]_q - [\alpha_{a_i}+\beta_{b_i}+1]_q)\right) \left(\prod_{i\notin\{a_1,\ldots,a_j\}} [\alpha_i+1]_q\right)\left(\prod_{i\notin\{b_1,\ldots,b_j\}} [\beta_i+1]_q\right)$$

$$C_{\alpha,\beta} = (m+n)! \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j}\left(\frac{1}{(\alpha_{a_i})!(\beta_{b_i})!} - \frac{1}{(\alpha_{a_i}+\beta_{b_i})!}\right)\right) \left(\prod_{i\notin\{a_1,\ldots,a_j\}} \left(\frac{1}{\alpha_i!}\right)\right)\left(\prod_{i\notin\{b_1,\ldots,b_j\}} \left(\frac{1}{\beta_i!}\right)\right)$$

$$Z_{\alpha,\beta}(s) = \sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1\le a_1<\cdots<a_j\le l(\alpha)\\ 1\le b_1<\cdots<b_j\le l(\beta)}} \left(\prod_{i=1}^{j}\binom{\alpha_{a_i}+s-1}{s-1}\binom{\beta_{b_i}+s-1}{s-1} - \binom{\alpha_{a_i}+\beta_{b_i}+s-1}{s-1}\right) \left(\prod_{i\notin\{a_1,\ldots,a_j\}} \binom{\alpha_i+s-1}{s-1}\right)\left(\prod_{i\notin\{b_1,\ldots,b_j\}} \binom{\beta_i+s-1}{s-1}\right)$$

$$\chi_{\alpha,\beta}(t) = t^{m+n-l(\alpha)-l(\beta)}(t-1)^{l(\alpha)+l(\beta)} \sum_{j=0}^{\min(l(\alpha),l(\beta))} \binom{l(\alpha)}{j}\binom{l(\beta)}{j}\left(\frac{1}{1-t}\right)^{j}$$

PROOF. This follows from direct application of Proposition 4.0.3, together with a small amount of additional simplification in the case of the characteristic polynomial formula. $\square$

Theorem 4.1.3 *Summing over compositions α and β indexed by monomials in non-commuting variables u_i and v_j satisfying relations $u_i v_j = v_j u_i$ for all $i, j > 0$, the following identities hold.*

$$\begin{aligned}
\sum_{\alpha,\beta} \Omega_{\alpha,\beta} u_\alpha v_\beta &= \frac{1}{1 - \sum_{k>0}(k+1)u_k - \sum_{l>0}(l+1)v_l + \sum_{k,l>0}(k+l-1)u_k v_l} \\
\sum_{\alpha,\beta} \Omega_{\alpha,\beta}(q) u_\alpha v_\beta &= \frac{1}{1 - \sum_{k>0}[k+1]_q u_k - \sum_{l>0}[l+1]_q v_l + \sum_{k,l>0}[k+l-1]_q u_k v_l} \\
\sum_{\alpha,\beta} \left(\frac{C_{\alpha,\beta}}{(m+n)!}\right) u_\alpha v_\beta &= \frac{1}{1 - \sum_{k>0}\left(\frac{1}{k!}\right) u_k - \sum_{l>0}\left(\frac{1}{l!}\right) v_l + \sum_{k,l>0}\left(\frac{1}{(k+l)!}\right) u_k v_l} \\
\sum_{\alpha,\beta} Z_{\alpha,\beta}(s) u_\alpha v_\beta &= \frac{1}{1 - \sum_{k>0}\binom{s+k-1}{s-1} u_k - \sum_{l>0}\binom{s+l-1}{s-1} u_l + \sum_{k,l>0}\binom{s+k+l-1}{s-1} u_k v_l} \\
\sum_{\alpha,\beta} \chi_{\alpha,\beta}(t) u_\alpha v_\beta &= \frac{1}{1 - \sum_{k>0} f(t,k) u_k - \sum_{l>0} f(t,l) v_l + \sum_{k,l>0} f(t,k+l) u_k v_l}
\end{aligned}$$

for $f(t,j) = t^{j-1}(t-1)$.

PROOF. To simplify notation, let $\alpha - \alpha_{a_i} = (\alpha_1, \ldots, \hat{\alpha}_{a_i}, \ldots, \alpha_k)$ and $\beta - \beta_{b_i} = (\beta_1, \ldots, \hat{\beta}_{b_i}, \ldots, \beta_l)$. The identities follow from the following recurrences for $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_l)$.

$$
\begin{aligned}
\Omega_{\alpha,\beta} &= (\alpha_k+1)\Omega_{\alpha-\alpha_k,\beta}+(\beta_l+1)\Omega_{\alpha,\beta-\beta_l}-(\alpha_k+\beta_l+1)\Omega_{\alpha-\alpha_k,\beta-\beta_l}\\
\Omega_{\alpha,\beta}(q) &= [\alpha_k+1]_q\Omega_{\alpha-\alpha_k,\beta}(q)+[\beta_l+1]_q\Omega_{\alpha,\beta-\beta_l}(q)-[\alpha_k+\beta_l+1]_q\Omega_{\alpha-\alpha_k,\beta-\beta_l}(q)\\
C_{\alpha,\beta} &= \binom{m+n}{\alpha_k}C_{\alpha-\alpha_k,\beta}+\binom{m+n}{\beta_l}C_{\alpha,\beta-\beta_l}-\binom{m+n}{\alpha_k+\beta_l}C_{\alpha-\alpha_k,\beta-\beta_l}\\
Z_{\alpha,\beta}(s) &= \binom{s+\alpha_k-1}{s-1}Z_{\alpha-\alpha_k,\beta}(s)+\binom{s+\beta_l-1}{s-1}Z_{\alpha,\beta-\beta_l}(s)\\
&- \binom{s+\alpha_k+\beta_l-1}{s-1}Z_{\alpha-\alpha_k,\beta-\beta_l}(s)\\
\chi_{\alpha,\beta}(t) &= \left(t^{\alpha_k-1}(t-1)\right)\chi_{\alpha-\alpha_k,\beta}(t)+\left(t^{\beta_l-1}(t-1)\right)\chi_{\alpha,\beta-\beta_l}(t)\\
&- \left(t^{\alpha_k+\beta_l-1}(t-1)\right)\chi_{\alpha-\alpha_k,\beta-\beta_l}(t)
\end{aligned}
$$

These recurrences may be proven in a similar fashion to Lemma 4.1.1. □

Question 4.1.1 *Do results of Simion and Stanley [SS, p.25-32] about the monoid of multiplicative functions generalize to generating functions in several variables, e.g. the variables u_i, v_j with relations $u_iv_j = v_ju_i$ for all positive integers i and j.*

4.2 k-shuffle posets

To some degree, this section mimics the arguments of the previous section. Consequently, our proofs are abbreviated, except where they differ in a significant way from the proofs for shuffle posets of multisets.

Recall our notation $\overline{\alpha}$ for the composition $(\alpha_1, \ldots, \alpha_{k-1})$ obtained from $\alpha = (\alpha_1, \ldots, \alpha_k)$ by deleting the last part. Let $F(S, \alpha^1, \ldots, \alpha^k)$ be F_P for the k-shuffle poset obtained from $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ by replacing $\alpha^{(j)}$ by $\overline{\alpha^{(j)}}$ for each $j \in S$.

Proposition 4.2.1 *The k-shuffle posets satisfy the recurrence*

$$F_{\alpha^{(1)},\ldots,\alpha^{(k)}} = \sum_{S\subseteq\{1,\ldots,k\}} (-1)^{|S|-1} F(S, \alpha^{(1)}, \ldots, \alpha^{(k)}).$$

PROOF. Recall that a chain is w_i-terminal if the last letter of w_i always occurs last in pairwise shuffled words in the chain that contain the letter. Proposition 2.2.2 shows that each chain is consistent with at least one shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$; in particular, this means that the chain is w_i-terminal for some nonempty collection of indices i which we call S. Such a chain will contribute to each summand on the right side which is indexed by any subset $T \subseteq S$. The coefficients for these summands are the Möbius functions of the boolean lattice of subsets of S, each multiplied by -1. Note that the empty set is the only subset of S not occuring, and $1 = \sum_{\emptyset \subset T \subseteq S} -\mu(\hat{0}, T)$, so each chain is accounted for exactly once. □

Let u_α denote the word $u_{\alpha_1} \ldots u_{\alpha_k}$ in Theorem 4.2.2.

Proposition 4.2.2 *The sum $\sum_{\alpha,\beta,\gamma} F_{\alpha,\beta,\gamma} u_\alpha v_\beta w_\gamma$ over all possible 3-tuples of compositions equals*

$$\left(1 - \sum_{i>0}(u_i + v_i + w_i)h_i + \sum_{i,j>0}(u_i v_j + u_i w_j + v_i w_j)h_{i+j} - \sum_{i,j,k>0} u_i v_j w_k h_{i+j+k}\right)^{-1}.$$

More generally, summing over all k-tuples of compositions yields

$$\sum_{\alpha^{(1)},\ldots,\alpha^{(k)}} F_{\alpha^{(1)},\ldots,\alpha^{(k)}} u^{(1)}_{\alpha^{(1)}} \cdots u^{(k)}_{\alpha^{(k)}} = \left(1 - \sum_{i=1}^{k} \sum_{j_1,\ldots,j_i>0} h_{j_1+\cdots+j_i} \sum_{1 \le t_1 < \cdots < t_i \le k} u^{(t_1)}_{j_1} \ldots u^{(t_i)}_{j_i}\right)^{-1}$$

where the expressions $F_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ are indexed by monomials in the alphabets $u^{(1)}, \ldots, u^{(k)}$ in noncommuting variables $u^{(i)}_j$ satisfying relations $u^{(i_1)}_{j_1} u^{(i_2)}_{j_2} = u^{(i_2)}_{j_2} u^{(i_1)}_{j_1}$. The monomial $u^{(i)}_{\alpha^{(i)}}$ is shorthand for $u^{(i)}_{\alpha^{(i)}_1} \ldots u^{(i)}_{\alpha^{(i)}_l}$, where $l = l(\alpha^{(i)})$.

PROOF. This follows from the recursive formula given in Proposition 4.2.1 just as in the case $k = 2$. □

The definition of $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ depends on the order of the compositions $\alpha^{(1)}, \ldots, \alpha^{(k)}$, so Greene asked if the rank generating function also depends on the order of the k words to be shuffled [Gr2]. The following implies that it does not, and furthermore that the flag f-vector does not.

Corollary 4.2.1 *If w_i is of type $\alpha^{(i)}$ for $1 \leq i \leq i$, then*

$$F_{\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(k)}} = F_{\alpha^{\sigma(1)},\ldots,\alpha^{\sigma(k)}}$$

for any $\sigma \in S_k$ permuting the k compositions specifying the types of the words to be shuffled.

PROOF. This is an immediate consequence of the symmetry of the right hand side of the formula given in Theorem 4.2.2 in the alphabets $u^{(1)}, \ldots, u^{(k)}$. □

Theorem 4.2.1 *The k-shuffle poset $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ is a lattice.*

PROOF. Let $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ be the k-shuffle poset given by words $w_1, \ldots, w_k$ where w_i is of type $\alpha^{(i)}$. We will construct $u \vee v$ in such a way that its minimality will be clear. We omit the argument for $u \wedge v$ because it is similar. Let us describe how to del-sert letters in u so as to obtain the smallest possible poset element which is also greater than v. We will specify which letters occur in which positions in each pairwise shuffled word assuming each letter in $w_1, \ldots, w_k$ occurs with multiplicity one; the reader may consult our proof for shuffle posets of multisets to find a convention to handle multiplicity that generalizes directly to k-shuffle posets.

We proceed by induction, describing how to del-sert letters belonging to w_i from u assuming that we are done del-serting letters belonging to $w_1, \ldots, w_{i-1}$ and have not yet del-serted any letters belonging to w_j for $j > i$. Any letter of w_1 that has been del-serted in v but not in u should be del-serted from u. If $u_1^c \sqcup\!\sqcup u_i$ is inconsistent with $v_1^c \sqcup\!\sqcup v_i$ for some $i > 1$, then there exists $a \in w_1$ and $b \in w_i$ such that a precedes b in $u_1^c \sqcup\!\sqcup u_i$ while b precedes a in $v_1^c \sqcup\!\sqcup v_i$, or vice-versa. In either case, we del-sert a from u. These are the only letters of w_1 to be del-serted from u.

Assume that we have del-serted letters belonging to $w_1, \ldots, w_{j-1}$ as necessary from u to obtain $\dot{u} \geq u$. We next del-sert from $\dot{u}$ any letter b of w_j which has been del-serted from v. We will call the result $\ddot{u}$. To do this, we need to specify where to insert b into $\dot{u}_i^c \sqcup\!\sqcup \dot{u}_j$ for each $i < j$. However, there is a unique way to do this which is consistent with $v_i^c \sqcup\!\sqcup v_j$, because $\dot{u}_i^c$ is a subword of v_i^c.

If $\ddot{u}_j^c \sqcup u_k$ is inconsistent with $v_j^c \sqcup v_k$ for some $k > j$, then again there exists $a \in w_j$ and $b \in w_k$ such that a precedes b in $\ddot{u}_j^c \sqcup u_k$ while b precedes a in $v_j^c \sqcup v_k$, or vice-versa. All such letters a belonging to w_j need to be del-serted from $\ddot{u}$. There is a unique position in which to insert a into $\ddot{u}_i^c \sqcup u_j$ for $i < j$. This is based on the position of b in $\ddot{u}_i^c \sqcup u_k$; namely, a must be inserted between the same two letters of $\ddot{u}_i^c$ that b is between in $\ddot{u}_i^c \sqcup u_k$. This is because $u \vee v$ must be consistent with a pairwise shuffled word in u which has a preceding b and also with a pairwise shuffled word in v which has b preceding a, or vice-versa, so a and b must occur in incomparable positions in $u \vee v$. By this algorithm, we construct $u \vee v$. □

Theorem 4.2.2 *The k-shuffle posets are EL-shellable.*

PROOF. Following Proposition 4.1.3, we label each edge with the letter to be del-serted. The labels need only be ordered in such a way that $a < b$ whenever $w(a) < w(b)$. □

This immediately implies the following.

Corollary 4.2.2 *The k-shuffle posets are Cohen-Macaulay.*

Corollary 4.2.3 *If w_i is of type 1^{m_i} for $1 \leq i \leq k$, then the k-shuffle poset given by words $w_1, \ldots, w_k$ satisfies*

$$\mu_P(\hat{0}, \hat{1}) = (-1)^{\mathrm{rkP}} \binom{m_1 + \cdots + m_k}{m_1, m_2, \ldots, m_k}.$$

For all other k-shuffle posets, $\mu_P(\hat{0}, \hat{1}) = 0$.

PROOF. Let n be the rank of P. Simply apply Proposition 4.0.1 and examine the coefficient of h_{1^n} in F_P. Alternatively, we may easily count the decreasing chains in our EL-labelling. These are in bijection with the distinct ways of shuffling $w_1, \ldots, w_k$ if each letter has multiplicity 1, and there are no decreasing chains otherwise. □

Corollary 4.2.4 *The k-shuffle posets have collapsible order complex, except when each letter occurs with multiplicity one. In this case, the distinct ways of shuffling the k-words index the cycles in a homology basis.*

PROOF. The Möbius function is the alternating sum of the ranks of the reduced homology groups, but these vanish except possibly in top dimension since the order complex is Cohen-Macaulay. Hence, the reduced homology groups all vanish when the Möbius function is 0. When these groups all vanish and the complex is shellable, this implies conllapsibility.

When each letter occurs with multiplicity one, then each way of shuffling the k words gives rise to a boolean sublattice, which in turn contributes a cycle to the top homology of the order complex. These cycles are indexed by the decreasing chains in the EL-labeling given in Proposition 4.2.2. □

Remark 4.2.1 *An NBB basis is given for the traditional shuffle posets in [BlS, p.106]. This generalizes in a natural way to k-shuffle posets in which each letter occurs with multiplicity one. The partial order on atoms is based on the letter to be del-serted. If an atom a_1 del-serts a letter of w_i while another atom a_2 del-serts a letter of w_j for $i < j$, then $a_1 \trianglelefteq a_2$, and otherwise the two atoms are incomparable. Each shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ gives rise to an NBB set consisting of all atoms which are consistent with this shuffled word; these give the whole basis of NBB sets.*

In fact, the k-shuffle posets are even supersolvable.

Theorem 4.2.3 *The k-shuffle posets are supersolvable.*

PROOF. Any saturated chain involving the empty word is an M-chain. This is because $W_{\alpha^{(1)},\dots,\alpha^{(k)}}$ may be decomposed into overlapping products of chains all of which include this chain, and because every saturated chain belongs to one of these product of chains sublattices. The rest follows similarly to the $k = 2$ case. □

Proposition 4.2.3 *The k-shuffle posets have symmetric chain decompositions.*

PROOF. Specializing the chain decomposition of Chapter 3 to poset elements considered as 1-chains yields a decomposition into symmetrically placed products of chains. This is similar to the proof for shuffle posets of multisets. Each shuffled word $w_1 \sqcup \cdots \sqcup w_k$ gives rise to a collection of descent blocks B, as discussed in the computation of F_P. Let $a_1, \ldots, a_j$ be the distinct letters in a descent block, ordered so that $w(a_i) < w(a_{i'})$ for $i < i'$. Let $m(a_i)$ be the multiplicity with which the letter a_i occurs. We index identical copies of a_i by $a_{i_1}, \ldots, a_{i_{m(a_i)}}$. Poset elements to be assigned to the piece of the decomposition specified by $w_1 \sqcup \cdots \sqcup w_k$ are those elements satisfying the constraints $a_{i_1} < a_{i-1_{m(a_{i-1})}}$ for $1 < i \leq j$ for each descent block in the word specifying this piece of the decomposition. These poset elements again form a symmetrically placed product of chains, because of symmetry in the constraints and because we are only considering poset elements in some product of chains P_w. □

One may obtain the rank generating function, characteristic polynomial and zeta polynomial for k-shuffle posets by expressing F_P in terms of complete symmetric functions (by way of the combinatorial definition of skew-Schur function of ribbon shape, as in the proof of Theorem 3.3.1) and then applying Proposition 4.0.3. The resulting formulas would be sufficiently unwieldy that we do not include them.

4.3 Noncrossing partition lattices for the classical reflection groups

Let us give two applications of the chain decompositions of Chapter 3 to the noncrossing partition lattices for the classical reflection groups.

Theorem 4.3.1 *The interpolating BD noncrossing partition lattices have symmetric boolean decompositions.*

PROOF. The point is to restrict the chain decomposition of Chapter 3 to 1-chains. This is made completely explicit by Theorem 5.4.3 and Theorem 5.3.2. The former gives a chain-labelling called an R^*S-labelling which induces what is known as a local symmetric group action on maximal chains. The chain decomposition of Chapter 3 restricted to maximal chains is a decomposition into the orbits of this action. On the other hand, Theorem 5.3.2 shows how to construct a symmetric boolean decomposition from an R^*S-labelling. Each poset element is assigned to a saturated chain containing it that comes as early as possible in our chain decomposition, so this amounts to assigning the element directly to the earliest piece of the chain decomposition containing it. □

Corollary 4.3.1 *The interpolating BD noncrossing partition lattices have symmetric chain decompositions.*

PROOF. This follows immediately from Theorem 4.3.1, together with the fact that products of chains have symmetric chain decompositions. □

Reiner previously constructed symmetric boolean decompositions for the type B noncrossing partition lattices and then separately for the interpolating BD noncrossing partition lattices [Re, p.16-18]. When we obtain symmetric boolean decompositions as specializations of chain decompositions, a single construction applies to all types at once.

The chain decomposition in Chapter 3 for type A noncrossing partition lattices also brought to our attention the existence of a saturated chain that belongs to each of the product of chains subposets in the decomposition and thereby turns out to be an M-chain.

Theorem 4.3.2 *The noncrossing partition lattice $NC^A(n)$ is supersolvable.*

PROOF. This is an easy consequence of our chain decomposition. Consider the saturated chain in the type A noncrossing partition lattice which has the partition with one nontrivial block $\{1, \ldots, i+1\}$ as its element of rank i. This is an M-chain because it lies in each of the products of chain subposets giving rise to our chain decomposition; the argument is then identical to our argument in Proposition 4.1.5 for shuffle posets of multisets. □

Björner showed that all supersolvable lattices have EL-labellings in [Bj2, p.166-167]. If $v_0 \prec \cdots \prec v_n$ is an M-chain, then an EL-labelling is defined by $\lambda(x, y) =$

$\min\{i|x \vee v_i = y \vee v_i\}$. We thus recover the EL-labelling of Björner-Gessel [Bj2, p.165] discussed in conjunction with a chain decomposition for noncrossing partition lattices in Chapter 3. In the noncrossing partition lattices for the other classical reflection groups types, the analogous chain no longer lies in all the product of chain subposets giving rise to our chain decomposition, and we are not aware of the existence of any M-chain.

Chapter 5

Local symmetric group actions on maximal chains

When the chain decompositions given in Chapter 3 are restricted to maximal chains, the result is often a decomposition into the orbits of what are known as local symmetric group actions on maximal chains.

Definition 5.0.1 *A symmetric group action on the maximal chains in a finite, ranked poset is* **local** *if the adjacent transpositions act in such a way that* $(i, i+1)$ *sends each maximal chain either to itself or to one differing only at rank* i.

In this chapter, we prove that if the symmetric group acts locally on a lattice, then each orbit considered as a subposet is a product of chains. We also show that all posets with local symmetric group actions induced by labellings known as R^*S-labellings have symmetric boolean decompositions. Furthermore, we indicate how any R^*S-labelling induces a chain decomposition into collections of chains of the form described by Lemma 3.1.1.

Section 5.4 provides an R^*S-labelling for the type B, D and interpolating BD noncrossing partition lattices. This generalizes the R^*S-labelling of Stanley for the traditional noncrossing partition lattices, given in [St5, p.7-10], and it answers a question of Stanley [St5, p.15]. Simion and Stanley also gave an R^*S-labelling for the posets of shuffles in [SS, p.10-15]; this is not possible for more general shuffle posets. By definition, every R^*S-labelling induces a local symmetric group action on maximal chains. The orbits of these induced actions on noncrossing partition lattices of all types and on traditional shuffle posets may alternatively be obtained by restricting the chain decompositions given in Chapter 3 to maximal chains, as we will discuss in Section 5.4.

Recall Stanley's observation that when F_P is a symmetric function then the number of maximal chains in P equals the dimension of the virtual symmetric group representation with F_P as Frobenius characteristic. To see this, note that the number of maximal chains is the coefficient of m_1^n in the monomial basis expansion for F_P, or equivalently the coefficient of p_1^n in the power-sum symmetric function expression for F_P; this coefficient is the character of the virtual representation evaluated at the identity, i.e. the dimension of the virtual representation. By similar reasoning, the number of maximal chains in P equals the dimension of the virtual symmetric group representation with ωF_P as Frobenius characteristic. Simion and Stanley show that symmetric group actions on maximal chains with Frobenius characteristic F_P are rare, but that any R^*S-labelling induces a symmetric group action on maximal chains with Frobenius characteristic ωF_P [SS, p.19-21].

5.1 Expressing poset structure in terms of rhombic tiling and flips

There is a correspondence between rhombic tilings of a region in the plane and equivalence classes of reduced expressions for a permutation up to commutation. This naturally translates symmetric group structure to poset structure when S_n acts locally on the maximal chains in a poset. We begin by reviewing this correspondence which is thoroughly examined in [El] because we will use it to explain why orbits of local symmetric group actions on lattices are always products of chains.

When a permutation w is written as a product of adjacent transpositions $w = s_{a_1}s_{a_2}\dots s_{a_l}$ with l as small as possible, such a product is called a **reduced expres-**

sion for w. To obtain from this a rhombic tiling, begin with a vertical path consisting of $n+1$ nodes; as one reads off each successive adjacent transposition s_{a_i} in a reduced expression, draw a new node to the right of the current node of rank a_i, and attach this new node to the nodes of rank $a_i \pm 1$ in the current path to obtain a new path. The resulting region is bounded on the left by the initial path, on the right by the

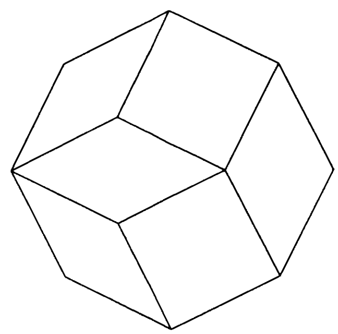

Figure 5-1: The rhombic tiling given by the reduced expression $s_1 s_3 s_2 s_3 s_1 s_2$

final path, and is tiled by quadrilaterals. These quadrilaterals may be replaced by rhombi by appropriately adjusting line segment slopes. Figure 5-1 gives an example of a rhombic tiling obtained in this manner from the reduced expression $s_1 s_3 s_2 s_3 s_1 s_2$. Two reduced expressions differing only by commutation relations give rise to the same rhombic tiling. Applying a braid relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ to a reduced expression amounts to a substitution within a tiling as in Figure 5-2. Consequently, any two

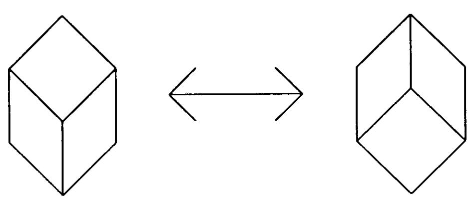

Figure 5-2: The Coxeter relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ in terms of tilings

reduced expressions for the same permutation give rise to rhombic tilings which fit in exactly the same planar region. One may obtain any rhombic tiling for a particular region from any other by applying braid relations.

A rhombic tiling also naturally records how a maximal chain is deformed under a local symmetric group action by successively applying the adjacent transpositions in a particular reduced expression for a permutation. If $p_2 = wp_1$, then each reduced expression for w gives rise to a (potentially distinct) way of deforming the maximal chain p_1 to p_2 within a poset. The structure of a poset with a local symmetric group action must allow for all possible ways of deforming one maximal chain to another.

Each rhombic tiling may be viewed as the projection of a discrete 2-dimensional surface S within a hypercube or multi-dimensional box onto a generic plane. Such a surface S may be deformed via braid relations (as in Figure 5-2) to surfaces coming from other reduced expressions for the same permutation; relations of the form $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ will take surfaces which include the front three faces of a cube to surfaces which instead includes the back three faces. The surfaces given by the same permutation will have the same boundary. The collection of rhombic tilings for a particular region gives rise to all the minimal discrete surfaces within a multi-dimensional box which have some fixed boundary. This point of view leads us to prove in Section 5.2 that the maximal chains in an orbit of a local symmetric group action must be arranged in such a way that they form the skeleton of such a multi-dimensional box. Otherwise, braid relations would be violated or an orbit would be incomplete (or both).

This does not, however, imply that each orbit is a product of chains since the nodes in such a skeleton need not all be distinct. In Section 5.2, we prove that the nodes are distinct when the poset is a lattice and conclude that the orbits in lattices are products of chains. In Section 5.3, we examine local actions induced by labellings known as R^*S-labellings. We prove that all posets with R^*S-labellings have symmetric chain decompositions.

5.2 A characterization of the orbits of local symmetric group actions on lattices

Simion and Stanley have shown in [SS] that the Frobenius characteristic of a local symmetric group action on an orbit is always a complete symmetric function. Theorem 5.2.2 will provide a more geometric proof of this result in order to show how orbits are realized within posets. We use this to characterize the orbits of local symmetric group actions on lattices in Theorem 4, answering a question of Stanley.

Figure 5-3 gives an example of how the situation differs between posets and lattices. When we identify the nodes labelled $(0, 3)$ and $(3, 0)$ within a product of two 4-chains, the resulting poset has a local symmetric group action with three orbits. One orbit consists of the maximal chains from the original product of chains before identification. Two new maximal chains are introduced by virtue of crossover being possible at the identified node. These maximal chains give rise to two trivial orbits, one of which is depicted by the shaded line in Figure 5-3.

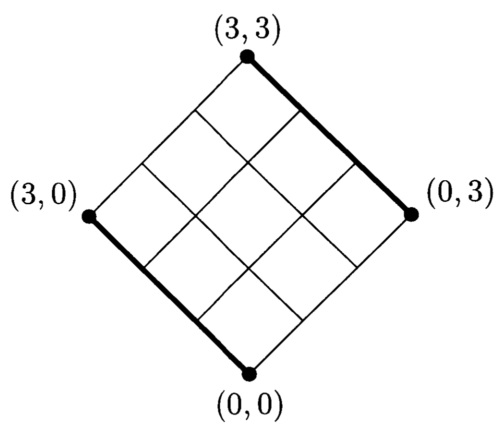

Figure 5-3: A product of chains with node identification

Note, however, that this example is not a lattice.

Theorem 5.2.1 *If S_n acts locally on the maximal chains in a poset, then the elements of an orbit subposet are the nodes in any maximal chain within the orbit specifying it. The covering relations are induced by covering relations from the maximal chains in the orbit.*

In lattices, the maximal chains in an orbit subposet turn out to be exactly the maximal chains belonging to the orbit specifying it.

The next lemma justifies geometric claims within the proof of Theorem 5.2.2.

Lemma 5.2.1 *If $s_i(p) \neq p$ and $s_{i+1}(p) = p$, then $s_{i+1}(s_i(p)) \neq s_i(p)$. Similarly, if $s_{i+1}(p) \neq p$ and $s_i(p) = p$, then $s_i(s_{i+1}(p) \neq s_{i+1}(p)$.*

PROOF. If $s_{i+1}(s_i(p)) = s_i(p)$ and $s_{i+1}(p) = p$, then

$$\begin{aligned} s_i(p) &= s_{i+1}(s_i(p)) \\ &= s_{i+1}s_i(s_{i+1}(p)) \\ &= s_i(s_{i+1}(s_i(p))) \\ &= s_i(s_i(p)) \\ &= p. \end{aligned}$$

The second assertion follows similarly. □

In Theorem 5.2.2 we will define a map ϕ from maximal chains in a poset to lattice paths in $\mathbb{Z}^n$. Lemma 5.2.1 implies that whenever $im(\phi)$ includes two lattice paths involving segments **abdf** and **acdf**, (in Figure 5-4) respectively, and otherwise agreeing, then $im(\phi)$ will also include a lattice path through **acef** which otherwise agrees with both these paths. No assumption is made about whether **bd** is perpendicular or parallel to **df**.

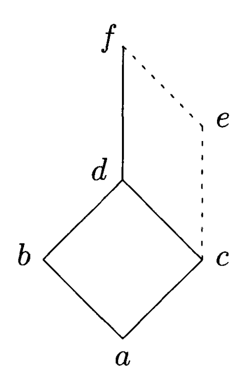

Figure 5-4: Building an orbit

Theorem 5.2.2 *If S_n acts locally on the maximal chains in a poset, then the Frobenius characteristic of the action is an h-positive symmetric function.*

PROOF. We prove that the S_n-action on any orbit is isomorphic to a local action on some product of chains $C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$ which has Frobenius characteristic h_λ.

We will refer to maximal chains in a poset P as P-chains. We claim that any orbit may be embedded by a map ϕ into the lattice $\mathbb{Z}^n$ in such a way that poset rank is encoded as sum of coordinates and P-chains are sent to lattice paths within $\mathbb{N}^n$. We will define ϕ in such a way that $im(\phi)$ will be the collection of all minimal lattice paths from the origin to a particular endpoint in $\mathbb{N}^n$. Furthermore, s_i will act nontrivially on a P-chain p whenever the segment of the lattice path $\phi(p)$ from rank $i-1$ to rank i is perpendicular to the segment from rank i to $i+1$. When path segments are labelled by lattice basis vectors, then applying an adjacent transposition will amount to swapping a lattice path with one with two consecutive labels transposed.

We define ϕ by choosing a P-chain p and specifying how to embed wp into $\mathbb{N}^n$ for each $w \in S_n$. The embedding will be based on a choice of reduced expression for w, but we will check that all reduced expressions for the same permutation w yield the same lattice path $\phi(wp)$. To conclude that ϕ is well-defined, we will also need to show that $\phi(w_1p) = \phi(w_2p)$ whenever $w_1p = w_2p$.

If $s_i(p) = p$ for all $i < a_1$ and $s_{a_1}(p) \neq p$, then let the lattice path $\phi(p)$ begin with a segment from $(0, \ldots, 0)$ to $(a_1, 0, \ldots, 0)$. The lattice path $\phi(p)$ will change direction at rank i for each i such that $s_i(p) \neq p$. In particular, if s_{a_2} acts nontrivially on p and all intermediate s_i act trivially on p, then $\phi(p)$ includes the segment from $(a_1, 0, \ldots, 0)$ to $(a_1, a_2-a_1, 0, \ldots, 0)$. At this point, we may specify how $\phi(wp)$ is related to $\phi(p)$ for any w which only involves the adjacent transpositions $s_1, \ldots, s_{a_2-1}$. If $s_jp_1 = p_2 \neq p_1$ for a P-chain p_1 which has already been embedded up to rank $j+1$, then p_2 is embedded up to rank $j+1$ by replacing the node of rank j in $\phi(p_1)$ with the only other node of rank j in $\mathbb{N}^n$ which together with the rest of $\phi(p_1)$ gives a lattice path. In this way, the embedding of p up to rank a_2 locally gives rise to every possible discrete path of minimal length from the origin to $(a_1, a_2 - a_1, 0, \ldots, 0)$; first one obtains $\phi(s_{a_1}(p))$, and repeated application of Lemma 5.2.1 yields all minimal length lattice paths from $(0, \ldots, 0)$ to $(a_1, a_2 - a_1, 0, \ldots, 0)$. These paths may be sequentially embedded in many different orders, but the commutation relations $s_i s_j = s_j s_i$ for $|j - i| \geq 2$ force all choices to be equivalent.

The direction in which to extend $\phi(p)$ to rank $a_2 + 1$ is determined by how s_{a_2} acts upon the P-chains with image under ϕ passing through the lattice point $(a_1, a_2 - a_1, 0, \ldots, 0)$ which also agree with $\phi(p)$ afterwards. The edge out of $(a_1, a_2 - a_1, 0, \ldots, 0)$ in $\phi(p)$ needs to be perpendicular to exactly those segments into $(a_1, a_2 - a_1, 0, \ldots, 0)$ which belong to lattice paths which are acted upon nontrivially by s_{a_2}, and which also include the given segment out of $(a_1, a_2 - a_1, 0, \ldots, 0)$.

Lemma 5.2.1 implies that at each step of the embedding of p, the next segment of $\phi(p)$ should be perpendicular to all but at most one of the lattice path edges leading into this new segment, so embedding is feasible. In this fashion we may define $\phi(p)$. Each time $\phi(p)$ changes direction, we repeatedly apply Lemma 5.2.1 just as we did at rank a_1 to obtain lattice paths of the form $\phi(wp)$. The relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ imply that when three consecutive segments of some $\phi(wp)$ are all perpendicular, six lattice paths result all belonging to $im(\phi)$, and the restriction of these lattice paths to the interval form the skeleton of a cube.

Repeated application of Lemma 5.2.1 and braid relations thus yields every minimal lattice path from the origin to the endpoint of $\phi(p)$ as the image of some P-chain, so ϕ will be onto. We need only show that any pair of distinct lattice paths $\alpha, \beta \in im(\phi)$ come from distinct P-chains to insure that ϕ is well-defined. Let v_1 and v_2 be nodes in $\mathbb{Z}^n$ where α and β first differ. There must also exist lattice paths $\gamma, \gamma' \in im(\phi)$ containing v_1 and v_2, respectively, which otherwise agree with each other. From the definition of ϕ it follows that γ and γ' are the images of distinct P-chains q, q' which satisfy $q' = s_i(q)$ for $i = rank(v_1)$. Hence, v_1 and v_2 must be the images of distinct poset elements of rank i, implying α and β are the images of distinct P-chains, so ϕ is well-defined. Our definition of ϕ insures that ϕ is injective, since $\phi(p) \neq \phi(wp)$ whenever $p \neq wp$.

The local S_n action on the orbit is thus an action well-known to have Frobenius characteristic h_λ, as desired. $\square$

The argument we present next was gleaned from a more complicated proof involving the correspondence between rhombic tilings and commutation classes of reduced expressions for a permutation.

Theorem 5.2.3 *If S_n acts locally on a lattice, then each orbit is a product of chains.*

PROOF.

We first prove that identifying nodes in a product of two chains leads to a poset which is not a lattice. After this, we show how to reduce the proof of the theorem to this case. We assume throughout that there is no node identification at rank 1, because we dealt with this possibility while proving ϕ was well-defined in Theorem 5.2.2.

Consider a product of two chains, each of rank r, in which $a = (r,0)$ is identified with $b = (0,r)$ and there is no node identification below rank r. Suppose this poset is a lattice. We use induction on j to show that $(j,1) \leq a$ for all $j < r$. As the base case, observe that that $(0,1) \leq a$ since $a = b = (0,r)$. If $(j,1) \leq (r,0)$ for some $j \geq 0$, then $(j,1) \vee (j+1,0) \leq (r,0)$ for $j+1 \leq r$. Since $(j+1,0) \leq (j+1,1)$ and $(j,1) \leq (j+1,1)$ and rank $(j+1,1) =$ rank $(j,1)+1$, note that $(j,1) \vee (j+1,0) = (j+1,1)$ in the poset. The definition of join thus implies $(j+1,1) \leq (r,0) = a$ whenever $(j,1) \leq (r,0)$ for $j+1 \leq r$. By induction, $(r-2,1) \leq a$, so $a \geq (r-2,1) \vee (r-1,0) = (r-1,1)$, a contradiction.

There is one somewhat subtle point to be addressed in the way we will show a poset is not a lattice by restricting to some subposet and showing this cannot be a lattice. When we assume a poset is a lattice, we need to be careful about whether the join of two subposet elements also lies in the subposet. In the above induction, we only deal with joins $a \vee b$ of rank one more than the rank of a and b, so this must be the join of a and b in any lattice containing the above as a subposet.

Now consider any product of chains with nodes a and b of rank r identified and with no node identification below rank r. Choose a maximal chain p_1 through the node a and a maximal chain p_2 through b (before identification), and then restrict attention to the nodes in some deformation of p_1 to p_2. We choose p_1 and p_2 so that the number of adjacent transpositions needed to deform p_1 to p_2 is as small as possible. If we let $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$, using the coordinates given by the product of chains structure, then p_1 and p_2 both contain the node $(\min(a_1, b_1), \ldots, \min(a_k, b_k))$ and agree below this node. Furthermore, a minimal deformation will only affect nodes above $(\min(a_1, b_1), \ldots, \min(a_k, b_k))$. The nodes above $(\min(a_1, b_1), \ldots, \min(a_k, b_k))$ in a minimal deformation will give rise to a product of two chains with a and b identified, and with no node identification below this.

This last observation follows from the fact that the coordinates which increase in travelling along the maximal chain p_1 between $(\min(a_1, b_1), \ldots, \min(a_k, b_k))$ and a are completely disjoint from the set of coordinates which increase in p_2 between $(\min(a_1, b_1), \ldots, \min(a_k, b_k))$ and b. An example is illustrated in Figure 5-5. The product of two chains comes from interspersing steps in the direction of p_1 with steps in the direction of p_2, while travelling from $(\min(a_1, b_1), \ldots, \min(a_k, b_k))$ to $(\max(a_1, b_1), \ldots, \max(a_k, b_k))$

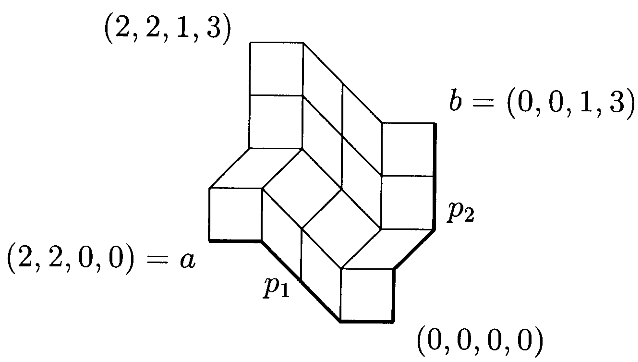

Figure 5-5: A 2-dimensional surface within a 4-dimensional product of chains

If a and b are identified in any product of chains, they will thus also be identified in a subposet which is a product of two chains, and so the original poset will not be a lattice. $\square$

5.3 Actions induced by chain-labellings

Simion and Stanley defined chain-labellings of posets, known as R^*S-labellings and RS-labellings, as part of a study of how the Frobenius characteristic of a local symmetric group action is related to the flag f-vector in [SS].

Recall that a chain labelling is an **R-labelling** if every rooted chain $\hat{0} \prec u_1 \prec \cdots \prec u_k = u \leq v$ has a unique extension to a saturated chain $\hat{0} \prec \cdots \prec v$ with weakly increasing labels on the interval from u to v. Similarly, a chain labelling is an **R*-labelling** if every rooted interval $\hat{0} \prec u_1 \prec \cdots \prec u_k = u \leq v$ has a unique extension to a saturated chain $\hat{0} \prec \cdots \prec v$ with strictly increasing labels on the interval from u to v. Note that the symmetric group acts on sequences of labels assigned to saturated chains by permuting the order of the labels, and this induces a local action on the maximal chains with corresponding labels. If a chain-labelling λ induces such a local symmetric group action on the maximal chains of a poset, and the sequences labelling the maximal chains are all distinct, then λ is an **S-labelling**. An S-labelling which is also R^* is an **R*S-labelling**, and an S-labelling which is also an R-labelling is an **RS-labelling**.

The following relationship between local symmetric group actions and flag f-vectors was proven in [SS].

Theorem 5.3.1 (Simion-Stanley) *If a poset P has an S-labelling, then let ϕ_P be the Frobenius characteristic of the induced local symmetric group action.*

1. *If P admits an RS-labelling, then $F_P = \phi_P = h_\lambda$ for some λ and P is the product of chains $C_{\lambda_1} \times \cdots \times C_{\lambda_k}$ where $\lambda = (\lambda_1, \ldots, \lambda_k)$.*
2. *If P admits an R^*S-labelling, then $F_P = \omega\phi_P$ where ω is the symmetric function involution which (in particular) swaps h_λ and e_λ.*

Theorem 5.3.1 of Simion and Stanley may be explained in terms of a chain decomposition. In defining this chain decomposition and in the subsequent construction of symmetric boolean decompositions, we will make reference to the unique saturated chain from u to v with increasing labels for any pair $u \leq v$. This is not well-defined for a chain-labelling which is not an edge labelling, but let us establish the following convention. When we refer to the unique increasing chain from u to v, we first choose the increasing maximal chain from $\hat{0}$ to u, and then based on this choice we select the resulting increasing saturated chain from u to v.

To obtain a chain decomposition from an R^*S-labelling, first extend each chain to a saturated chain by choosing the unique increasing saturated chain on each interval. Now assign the chain to the orbit which contains this saturated chain. Recall from earlier in the chapter that the Frobenius characteristic of each orbit of a local symmetric group action is a complete symmetric function h_λ and that the action is isomorphic to an action on a product of chains $C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$. According to Lemma 3.1.1, this orbit will contribute e_λ to F_P in the decomposition we have just given because we assign to the orbit those chains satisfying the appropriate set of strict inequalities.

Next we prove next that an R^*S-labelling induces a symmetric chain decomposition.

Theorem 5.3.2 *If a finite, ranked poset admits an R^*S-labelling, then the elements may be decomposed into a disjoint union of symmetrically placed boolean lattices.*

Proof. We define a map ϕ from elements of a finite poset P to symmetrically placed boolean lattices in the poset and show that this map is a decomposition. Let λ be an R^*S-labelling for a poset P of rank n. For each $v \in P$, there are unique saturated chains $\hat{0} = u_0 \prec u_1 \prec \cdots \prec u_k = v$ and $v = v_0 \prec \cdots \prec v_l = \hat{1}$ with strictly increasing labels. Since λ is an S-labelling, there exist u and w such that $\hat{0} \le u \le v \le w \le \hat{1}$ and rank $w = n-$ rank u with u and w satisfy the following two conditions: first, the set of labels on the unique rising chain from $\hat{0}$ to u is the same as the set of labels on the unique rising chain from w to $\hat{1}$. Second, the set of labels on the rising chain from u to v is completely distinct from the set of labels on the rising chain from v to w; both sets are completely distinct from the set of labels on the rising chains from $\hat{0}$ to u and from w to $\hat{1}$. This is possible by restricting S_n to acting locally on the saturated chain from $\hat{0}$ to v and likewise on the saturated chain from v to $\hat{1}$ to obtain new saturated chains with all common labels shifted down to below u and up to above w. There is a symmetrically placed boolean lattice $B_{u,w}$ on the interval from

u to w. It consists of all nodes reached by restricting S_n to acting locally on the orbit within this interval (u, w) which includes the increasing chain from u to w. Since λ is an R^*-labelling, u and w are uniquely specified, and $v \in B_{u,w}$, so let $\phi(v) = B_{u,w}$.

Note that if $\phi(v_1) = B_{u,w}$ and $v_2 \in B_{u,w}$, then $\phi(v_2) = B_{u,w}$, because the unique increasing chains from $\hat{0}$ to v_2 and from v_2 to $\hat{1}$ may be obtained by taking a maximal chain which includes v_2 in addition to u and w since v_2 belongs to $B_{u,w}$, and then applying a sequence of adjacent transpositions permuting the labels above v_2 and below v_2 separately. Hence, ϕ provides a decomposition. $\square$

Note that R^*S-labellings restrict to intervals, so Theorem 5.3.2 also applies to all the intervals in posets with R^*S-labellings.

Corollary 5.3.1 *If a finite, ranked poset admits an R^*S-labelling, then it has a symmetric chain decomposition.*

Proof. Theorem 5.3.2 provides a decomposition into symmetrically placed boolean lattices, and each of these has a symmetric chain decomposition. One may find an explicit construction of a symmetric chain decomposition for the boolean lattices in a survery article by Greene and Kleitman [GK], and this article also gives original references (de Bruijn et al., Leeb). $\square$

5.4 Specialization of chain decompositions into orbits

This section gives an R^*S-labelling for the type B, D and interpolating BD noncrossing partition lattices, answering a question raised by Stanley in [St5, p.15], and then briefly discusses R^*S-labellings for other posets. The R^*S-labelling for type B is closely related to the following labelling by parking functions for the traditional (type A) noncrossing partition lattice given in [St5, p.7-10].

Theorem 5.4.1 (Stanley) *If $u \prec v$ then there exist two distinct blocks B_1 and B_2 in the partition u which are merged in v. Without loss of generality, assume*

$\min B_1 < \min B_2$. *Let* $\lambda(u,v) = \max\{b \in B_1 | b < b' \text{ for all } \mathrm{b}' \in \mathrm{B}_2\}$. *Then* λ *is an* R^*S*-labelling for the noncrossing partition lattices.*

We define an edge-labelling λ for the type B noncrossing partition lattice, also in terms of covering relations. If v is obtained from u by merging C with $-C$ to form C_0, then C is entirely contained in some semicircle. In this case, we let $\lambda(u,v) = i$ where $i \in \{1,\ldots,n\}$ and $\pm i$ is the last element of C encountered while travelling clockwise about such a semicircle. If v is obtained from u by merging two components C_1 and C_2 (and simultaneously merging $-C_1$ with $-C_2$), then one of the following conditions is true (for some choice of which component is C_1 and which is C_2). Either there is some pair of elements both in C_2 such that all elements of C_1 lie between these two elements of C_2, or there is some semicircle containing both C_1 and C_2 such that all of C_2 comes before all of C_1 travelling clockwise about this semicircle. In either case, there is some $i \in \{1,\ldots,n\}$ such that $\pm i$ is the last element of C_2 encountered before the first clockwise element of C_1, and then we let $\lambda(u,v) = i$.

For example, listing only blocks with more than one element, the maximal chain

$$\emptyset \prec \pm\{5,6\} \prec \pm\{5,6\},\{\pm 4\} \prec \pm\{1,3\},\pm\{5,6\},\{\pm 4\} \prec \pm\{1,2,3\},\pm\{5,6\},\{\pm 4\}$$
$$\prec \{\pm 1,\pm 2,\pm 3,\pm 4\},\pm\{5,6\} \prec \{\pm 1,\pm 2,\pm 3,\pm 4,\pm 5,\pm 6\}$$

is labelled with the parking function $\lambda = (5,4,1,1,4,4)$. Figure 5-6 depicts how components are sequentially merged in this maximal chain; arc labels indicate the order in which components are merged.

We follow Stanley [St3,p.14] in referring to sequences in $\{1,\ldots,n\}^n$ as type B parking functions. The number of maximal chains in $NC^B(n)$ is n^n, and Theorem 5.4.2 will show that λ labels each maximal chain with a distinct type B parking function.

Theorem 5.4.2 *The labelling* λ *on the type B noncrossing partition lattice is an* R^*S*-labelling.*

Proof. We prove that λ is a bijection between maximal chains and sequences in $\{1,\ldots,n\}^n$ by recursively defining the inverse map. After that, we verify that the

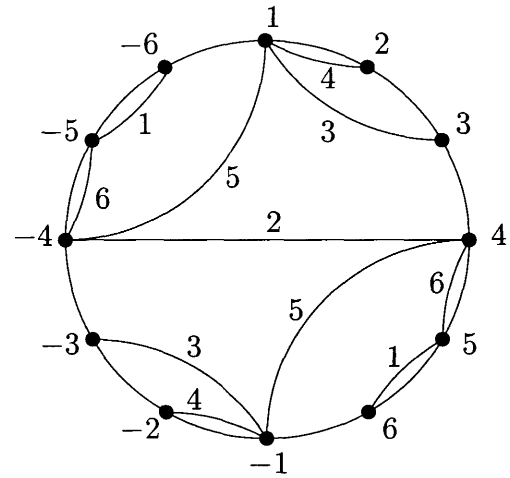

Figure 5-6: A maximal chain in a type B noncrossing partition lattice

S_n-action permuting the order of the digits in parking functions induces a local action on corresponding maximal chains to conclude that λ is an S-labelling. Finally, we check that λ is also an R^*-labelling.

It will suffice to provide the inverse to λ for those maximal chains in which 1 is merged with some $j \in \{2, \ldots, n, -1\}$ at rank 1. These chains will be labelled by exactly those parking functions with 1 as their first digit. The circular symmetry of the type B noncrossing partition lattice and of our labelling λ will allow us to conclude from this case that λ is a bijection in general. Given a parking function with first digit 1, the choice of which j to merge with 1 at rank 1 depends on the content (but not the order) of the remaining $n-1$ digits in the parking function. To determine j, let us rearrange the remaining labels in increasing order and view this weakly increasing sequence as a path from $(1,1)$ to (n,n) by steps of length 1 up and to the right. If every lattice point in this path is of the form (k,i) for $i \le k$, then let $j=-1$. Otherwise, let j be the unique integer such that $(j+1,j)$ belongs to the lattice path and this is the first place the path goes above the staircase path. (Lattice paths are used in a similar fashion in [Re, p.24].)

Of these $n-1$ digits specifying such a path, those which are less than j determine how elements between 1 and $j-1$ will be merged while digits between j and n determine how to merge elements between j and -1 on the circle. Note that the former correspond to type A parking functions on $1, \ldots, j-2$ while the latter correspond to elements of $[j, \ldots, n]^{n-j}$. Hence, the $n-1$ remaining digits give rise to a type A parking function on $1, \ldots, j-2$ interspersed with a type B parking function on $j, \ldots n$.

There is a corresponding decomposition of the space of maximal chains which recursively completes the bijection. This involves the following product structure which is also discussed in [Re, p.7], and which generalizes similar structure for type A as studied in [SU, p.]. Let $M_i(NC^B(n))$ denote the collection of maximal chains in the type B noncrossing partition lattice on $\pm 1, \ldots, \pm n$ which begin by merging $i \in \{1, \ldots, n\}$ with any $j \in \{i+1, \ldots, -i\}$ and merging $-i$ with $-j$. Let $M_{i,j}(NC^B(n))$ denote the restriction to maximal chains with a particular choice of j. Observe that $M_i(NC^B(n)) = \coprod_{j \in \{i+1, \ldots, -i\}} M_{i,j}(NC^B(n))$. Also note that the interval $(u, \hat{1})$ from a type B noncrossing partition u of rank 1 up to $\hat{1}$ is isomorphic to $NC^A(j-i) \times NC^B(n-j+i)$ if i is merged with j in u. Hence, maximal chains in such an interval are labelled by type A parking functions interspersed with type B parking functions, as desired.

To show that λ induces a local symmetric group action, we need to check that the maximal chain labelled by the type B parking function $(a_1, \ldots, a_n)$ differs only at rank i from the maximal chain labelled by $(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)$ for $a_i \neq a_{i+1}$. Note that λ^{-1} "decides" which node to visit next at each step in choosing a maximal chain from $\hat{0}$ to $\hat{1}$ based only on the content of the remaining digits. This implies that the two maximal chains will agree up to rank $i-1$. Also observe that merge steps i and $i+1$ "commute" for $a_i \neq a_{i+1}$ by virtue of the recursively defined bijection λ. This notion of commutativity comes from treating merge steps as operators which take poset elements of rank $i-1$ to ones of rank i. Our discussion of edge-labelled graphs immediately following this theorem should clarify this point. The symmetric group relations are automatically satisfied since the action on maximal chains is induced by a valid symmetric group action on sequences of labels.

Finally, we claim that this S-labelling is also R^*. First note that the unique rising chain from $\hat{0}$ to $\hat{1}$ involves merging $\{1, \ldots, i\}$ with $\{i+1\}$ at stage i for $1 \leq i < n$ and then merging $\{1, \ldots, n\}$ with $\{-1, \ldots, -n\}$ at stage n. For $u \leq v$ the increasing chain from u to v is found similarly, but skipping steps merging components which are already merged in u or still not merged in v. □

One may associate graphs to orbits and edge labellings of these graphs to maximal chains within each orbit to make the recursive structure in the argument explicit. Figure 5-6 is an example of such a graph. Begin with a circle with nodes $1, \dots, n, -1, \dots, -n$ placed sequentially about it. For each covering relation $u \prec v$ in a maximal chain, we will draw a pair of arcs which are each labelled with the rank of v. For convenience, we will refer to left and right endpoints of arcs, by which we will mean the endpoint that appears to the left or right from the point of view from the center of the circle. The absolute value of the left endpoints of the pair of arcs labelled i will be the ith digit of the parking function which labels our maximal chain. Labelling poset edges with the absolute value of the right endpoints of the same arcs gives another poset edge-labelling. This labelling restricts to an EL-labelling for the type A noncrossing partition lattice; this type A labelling is due to Gessel and Björner [Bj2, p.165] and was studied by Edelman and Simion in [ES]. Unfortunately, the type B labelling is not also an EL-labelling. The type A analogue of edge-labelled graphs are equivalent to the vertex-labelled trees discussed in [ES, p.109-113].

If C is merged with $-C$ in the covering relation $u \prec v$, recall that there is some semicircle containing all of C. Draw an arc from i to $-i$ where i is the last element of C encountered travelling clockwise about this semicircle. If C_1 is merged with C_2 and C_1 lies entirely between consecutive elements of C_2, then draw an arc with left endpoint at the element of C_2 which comes last clockwise before reaching C_1. The right endpoint will be the first element of C_1 encountered continuing clockwise from this element of C_2. Otherwise, there is a semicircle which includes all of C_1 and all of C_2. Draw an arc connecting the nearest elements of C_1 and C_2 to each other.

These edge-labelled graphs always satisfy the following two conditions. First, each point on the circle is the right endpoint to exactly one arc. Second, the labels on the arcs with a particular left endpoint increase as one reads away from the center of the circle, i.e. as the right endpoints of these arcs progress counterclockwise.

Note that whenever two consecutive digits a_i and a_{i+1} in a parking function differ, the arcs labelled i and $i+1$ will have different left endpoints, so swapping these arc labels gives an edge-labelled graph which still satisfies the above two conditions. Hence, the two maximal chains with parking functions $(a_1, \ldots, a_n)$ and $(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)$ have edge-labelled graphs with these arc labels swapped, so the chains differ only at rank i. Successively applying adjacent transpositions shows that maximal chains p and wp in the same orbit give rise to the same underlying graph, but with arc-labels permuted by w.

One may often obtain R^*S-labellings for subposets by restriction of the labelling on the whole poset. This may be done with noncrossing partition lattices of type B by forbidding particular arcs within these circular graphs. The interpolating BD noncrossing partition lattices are an example of such a restriction.

Theorem 5.4.3 *The labelling λ restricts to an R^*S-labelling for the interpolating BD noncrossing partition lattices.*

PROOF. The maximal chains of type B which do not occur in a particular interpolating BD noncrossing partition lattice constitute entire orbits. Hence the restriction of the R^*S-labelling is still an S-labelling. An increasing chain between two partitions which both satisfy $C_0 \neq \{\pm i\}$ never involves merging $\pm i$ to form $C_0 = \{\pm i\}$ at an intermediate step, so every remaining interval still has a unique increasing chain. □

Simion and Stanley proved in [SS, p.19-21] that $F_P = \omega\phi_P$, where ω is the symmetric function involution which (in particular) swaps e_λ with h_λ. Thus, one may recover F_P for interpolating BD noncrossing partition lattices from the fact that $\phi_P = \omega F_P$, so this yields another proof of Corollary 3.4.2. Let us recall the formula and show how it may be easier to restrict R^*S-labellings than to restrict such things as symmetric chain decompositions which follow from these labellings.

Corollary 5.4.1 *If P is the interpolating BD noncrossing partition lattice NC_S^{BD}, then*

$$F_P = \sum_{\substack{\alpha \in \mathbb{N}^n \cap PF_S \\ \alpha_1 + \cdots + \alpha_n = n}} e_\alpha$$

for $PF_S = \bigcap_{i\in S}\{\alpha|\alpha_i + \cdots + \alpha_j < j - i + 1$ *for some* $i \le j \le n$ *or* $\alpha_i + \cdots + \alpha_n + \alpha_1 + \cdots + \alpha_j < j + n - i + 1$ *for some* $j < i\}$.

PROOF. Theorem 5.4.3 implies $NC_S^{BD}(n)$ has an R^*S-labelling, so we again use the fact that $F_P = \omega\phi_P$. Note that the type B orbits which survive are those with edge-labelled graph not involving an arc from i to $-i$ for any $i \in S$. The definition of PF_S allows the parking functions corresponding to exactly these maximal chains. □

For example, if $P = NC_S^{BD}(3)$ and $S = \{1\}$, then $F_P = e_1^3 + 5e_1e_2 + 2e_3$. This poset has two fewer orbits than $NC^B(3)$, since the orbits of $NC^B(3)$ with label content $(1,1,1)$ and $(1,1,2)$ are no longer permitted.

In the specialization to type B, we have $S = \emptyset$, which means $\{1,\ldots,n\}^n \cap PF_S = \{1,\ldots,n\}^n$, and we recover Corollary 5.4.1. When $S = \{1,\ldots,n\}$, i.e. the type D specialization, we obtain $F_P = \sum_{a\in PF_{\{1,\ldots,n\}}} e_{\alpha(a)}$. Using recursive product structure, this can be reformulated as

$$F_P = F_{NC^B(n)} - n\left(\sum_{T\subseteq\{3,\ldots,n\}} e_{|T|+2}F_{NC^A(t_1-2)}F_{NC^A(n-t_{|T|})}\left(\prod_{i=1}^{|T|-1} F_{NC^A(t_{i+1}-t_i)}\right)\right).$$

Each of the subtracted sums comes from forbidding a particular 0-block of the form $\{\pm i\}$. The edge-labelled graphs involving an arc from i to $-i$ also have arcs from i to $i+1$, so T specifies the right endpoints of all other arcs with left endpoint i. We choose $T \subseteq \{3,\ldots,n\}$ above for the case $i = 1$, but the contribution to F_P will be the same for any i, so we multiply by n to consider all possible $i \in \{1,\ldots,n\}$. This n may be replaced by any $j \in \{1,\ldots,n\}$ to give a similar formula for an interpolating BD noncrossing partition lattice with $j = |S|$.

Remark 5.4.1 *Shuffle posets of multisets with repetition of letters, k-shuffle posets and graded monoid posets do not in general have R^*S-labellings.*

The fact that F_P is Schur-positive for shuffle posets of multisets would suggest the possibility that F_P or ωF_P might be the Frobenius characteristic of a symmetric group action permuting maximal chains., However, the coefficient of h_1^n in F_P would count the number of orbits in such an action with Frobenius characteristic ωF_P, and this coefficient is 0 for shuffle posets of multisets once multiplicity is introduced, making such an action impossible. This coefficient is 1 in ωF_P, making such an action with Frobenius characteric F_P impossible also because the number of maximal chains does not divide the size of the symmetric group that would act on them.

Question 5.4.1 *Is there a local group algebra action on shuffle posets of multisets with Frobenius characteristic F_P or ωF_P (or a function related to F_P in some interesting way)? Does the representation with Frobenius characteristic F_P or ωF_P act on the vector space with basis given by the maximal chains in an interesting way?*

The k-shuffle do have a labelling which is nearly an R^*S-labelling; this labelling shares enough features with an R^*S-labelling that the arguments that posets with R^*S-labellings have symmetric chain decompositions and e-positive flag f-vectors will still apply. Unfortunately, two distinct saturated chains may be labelled in exactly the same way, so the labelling cannot be an S-labelling, but it will have all the other properties of an R^*S-labelling. We call such a labelling which lacks this one property an $\mathbf{R^*S^-}$**-labelling**.

Proposition 5.4.1 *If each letter occurs with multiplicity one in a k-shuffle poset $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$, then $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ admits an R^*S^--labelling.*

PROOF. This is similar to the chain-labelling of Simion and Stanley for traditional shuffle posets [SS, p.10-12]. That is, the edges coming from del-serting letters in any particular descent block will all be given identical labels. They are labelled with the letter in the descent block which occurs earliest in the chain, namely the one belonging to w_i for i as large as possible. We may use any total order on the letters as long as $a \in w_i$ is smaller than $b \in w_j$ for $i < j$.

Each interval then has a unique strictly increasing chain, because when we del-sert letters from w_i before w_j for each $i < j$, then there is no choice of insertion position to make. Since letters are not allowed to occur with multiplicity, such a chain is strictly increasing rather than weakly increasing. The labelling is an S^--labelling because given any two consecutive edges with distinct labels, the least shuffled word containing a saturated chain with both these edges does not have the letters as part of a single descent block, so the order of del-sertion may be reversed without changing the descent block to which either letter belongs. □

Graded monoid posets cannot in general have R^*S-labellings. One might expect there to be an R^*S-labelling at least in the squarefree case if there is a quadratic Gröbner basis, but even this is not always true. For example, this fails for Example 3.5.1. The only case we know of where there is nearly an R^*S-labelling is the squarefree case with a quadratic Gröbner basis which has what we call **leading term transitivity (LTT)**. Namely, we require that whenever the quadratic Gröbner basis has leading terms ab and bc, then it also has leading term ac. In this case, there will be an R^*S^--labelling.

Proposition 5.4.2 *If P is a graded monoid poset interval with only squarefree factorizations, and if there is a quadratic Gröbner basis B for I_Λ, and if B has property (LTT), then P has an R^*S^--labelling.*

PROOF. Each edge is labelled with the monoid element that is being added, except in the following situation. If the Gröbner basis B has a leading term uv where v is being added and u is a label that occured earlier in the chain, then the edge is instead labelled u. This will ensure that the labelling is R^* since each interval has a unique factorization which is completely reduced by the Gröbner basis. If B has leading terms uv and uw and if both v and w occur in a saturated chain before u occurs, then B will also have leading term vw by the (LTT) condition, so the edges in which v and w occured will both have the same label; thus, this label is also given to the edge in which u occurs. If two consecutive edges (v_{i-1}, v_i) and (v_i, v_{i+1}) in a saturated chain have distinct labels, then by definition of our labelling, there will be exactly one other saturated chain which has these labels reversed and which differs only at rank i from the chain, so the labelling will be an S^--labelling. □

If B has leading terms uv, uw and vw, then the iterval $[\hat{0}, uvw]$ has two distinct chains labelled uuu, so the above labelling is not always an S-labelling.

Chapter 6

Partitions of a multiset and lexicographic shellability

In this chapter, we generalize the partition lattice Π_n by removing the assumption that all the letters to be partitioned need to be distinguishable. A result of Ziegler implies that the order complexes of the resulting multiset partition posets are not always Cohen-Macaulay [Zi, p.218]. In Section 6.1, we define closely related cell complexes, and Section 6.4 proves that these cell complexes which we call refinement complexes are always shellable. Section 6.2 provides formulas for the Euler characteristic of refinement complexes, one of which generalizes the fact that $\mu(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$ for Π_n. In order to prove shellability of refinement complexes, we extend the notion of lexicographic shellability in two directions:

1. To pure balanced cell complexes with cells that resemble simplices, in a sense that we will make precise later.
2. To posets with chain-labellings such that:
   (a) each interval has a unique saturated chain with no codimension one intersections with earlier chains in the interval
   (b) each such chain is lexicographically smallest on its interval

The pure balanced cell complexes we study differ from simplicial complexes only in that we will allow a single set of vertices to specify more than one face. The main subtlety in defining what it means for such a pure balanced cell complex to have a lexicographic shelling is that we will need to specify an edge rather than a pair of vertices to determine an interval. This distinction leads to somewhat surprising results. The second generalized notion of shelling mention above turns out to be equivalent to what Kozlov calls CC-shellability, although he works primarily in terms of a different formulation of this condition [Ko]. Section 6.3 makes precise these two notions of lexicographic shellability and verifies that they have appropriate topological implications. Included throughout the chapter are numerous open questions, many of which we hope to explore soon. Much of the work in sections 6.2-6.4 is recent joint work with Robert Kleinberg.

6.1 Deformed Möbius functions and refinement complexes

We generalize the lattice Π_n of partitions of the set $\{1, \ldots, n\}$ to posets of partitions of multisets. Let λ be a partition of n, and let k be the number of parts in λ. Given the multiset $\{a_1^{\lambda_1}, \ldots, a_k^{\lambda_k}\}$, i.e. the multiset with the letter a_i occuring with multiplicity λ_i, we define a **multiset partition poset** to consist of the partitions of some multiset $\{a_1^{\lambda_1}, \ldots, a_k^{\lambda_k}\}$ reverse ordered by refinement. Let $\hat{0}$ be the partition into blocks of size one, and let $u \prec v$ if v is obtained from u by merging two blocks. When $\lambda = 1^n$, we recover the partition lattice Π_n, while $\lambda = (n)$ yields the poset of partitions of the integer n, again reverse ordered by refinement. The lattice Π_n is quite well behaved, while the poset of partitions of the integer n reverse ordered by refinement is less well-understood and in many ways less well-behaved. In particular, Ziegler showed that the poset of partitions of $\{a^n\}$ is not Cohen-Macaulay for $n \geq 19$ in [Zi, p. 218].

Example 6.1.1 (Ziegler) *The structure of the interval between* $a^6|a^5|a^3|a^2|a^2|a$ *and* $a^8|a^7|a^4$ *in the poset of partitions of* $\{a^{19}\}$ *prevents the poset from being Cohen-Macaulay. Simply note that* $a^8|a^7|a^4$ *may be decomposed in two different ways into* $a^6|a^5|a^3|a^2|a|a$, *given by* $8 = 6+2, 7 = 5+2, 4 = 3+1$ *and by* $8 = 5+3, 7 = 6+1, 4 = 2+2$. *This results in a face* F *given by any saturated chain from* $\hat{0}$ *to* $a^6|a^5|a^3|a^2|a^2|a$ *together with any saturated chain from* $a^8|a^7|a^4$ *to* $\hat{1}$ *such that* $lk(F)$ *is a disconnected graph.*

Figure 6-1 depicts the image in the order complex of a pair of chains belonging to this interval.

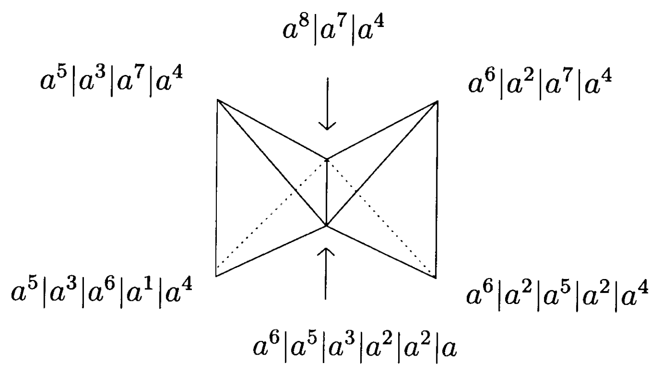

Figure 6-1: Two solid tetrahedra sharing an edge

The poset of partitions of $\{a^n\}$ is shellable for $n \leq 10$ [Zi, p.218]. We would like to better understand to what extent posets of partitions of multisets share the nice topological and combinatorial properties of Π_n. While the order complexes of multiset partition posets are not always Cohen-Macaulay, we will show that related cell complexes which we call refinement complexes are shellable, which implies they are Cohen-Macaulay. These complexes are similar enough to order complexes that we hope this analysis may be helpful in studying the topology of the order complexes themselves. Unlike Π_n, not all multiset partition posets are lattices, so this restricts the collection of tools at our disposal for studying them.

Question 6.1.1 *Is the order complex of every multiset partition poset has the homotopy type of a wedge of spheres concentrated in top dimension?*

Recall that the Möbius function of the partition lattice Π_n satisfies $\mu_P(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$ and that μ is multiplicative in the following sense: if $B_1, \dots, B_j$ are blocks of sizes $s_1, \dots, s_j$, respectively, then $\mu_{\Pi_{s_1+\cdots+s_j}}(\hat{0}, B_1|\dots|B_j) = \prod_{i=1}^{j} \mu_{\Pi_{s_i}}(\hat{0}, B_i)$. Unfortunately, the Möbius function for posets of multiset partitions is not always multiplicative. If we denote by $\mu(\hat{0}, a_1^{n_1}\dots a_k^{n_k})$ the Möbius function $\mu(\hat{0},\hat{1})$ for the poset of partitions of $\{a_1^{n_1}, \dots, a_k^{n_k}\}$, then for example $\mu(\hat{0}, ab) = -1$ while $\mu(\hat{0}, ab|ab) = 0 \neq (-1)^2$.

An application (discussed in Chapter 3) led us to examine a closely related function on multiset partition posets which is forced to be multiplicative and which we denote by μ'. Given a partition into blocks $B_1, \dots, B_j$, we define $\mu'(\hat{0}, B_1|\dots|B_j)$ to equal $\prod_{i=1}^{j} \mu'(\hat{0}, B_i)$ and for each individual block B_i we let $\mu'(\hat{0}, B_i) = -\sum_{\hat{0}\leq u<B_i} \mu'(\hat{0}, u)$. We will prove in Section 6.2 that

$$\mu'(\hat{0}, ab_1^{n_1}\dots b_k^{n_k}) = (-1)^{n-1}\frac{(n-1)!}{\prod_{i=1}^{j}(n_i)!}$$

when $n_1+\cdots+n_k = n-1$ and there is a distinguished letter a occuring with multiplicity one. This generalizes the result $\mu_P(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$ for the partition lattice Π_n since $\mu' = \mu$ when the letters being partitioned are all distinguishable. Note that $(-1)^{n-1}\frac{(n-1)!}{\prod_{i=1}^{j}(n_i)!}$ is not always an integer when none of the letters occur with multiplicity one, so it is too much to ask for this formula to hold for all multisets.

Let us interpret μ' as the reduced Euler characteristic of a cell complex which we call the refinement complex. First recall that the Möbius function $\mu_P(\hat{0},\hat{1})$ is the alternating sum $\sum_{i\geq 0}(-1)^i c_i$ in which c_i counts i-chains in $P - \{\hat{0},\hat{1}\}$; this is the reduced Euler characteristic of the order complex. Gian-Carlo Rota develops this beautiful and powerful connection between the Möbius function of a poset and the reduced Euler characteristic of its order complex in [Ro].

Definition 6.1.1 *Given an i-chain of comparable poset elements, one obtains from this an* **i-refinement sequence** *by also specifying at each stage of the refinement which partition blocks are split into which types of pieces.*

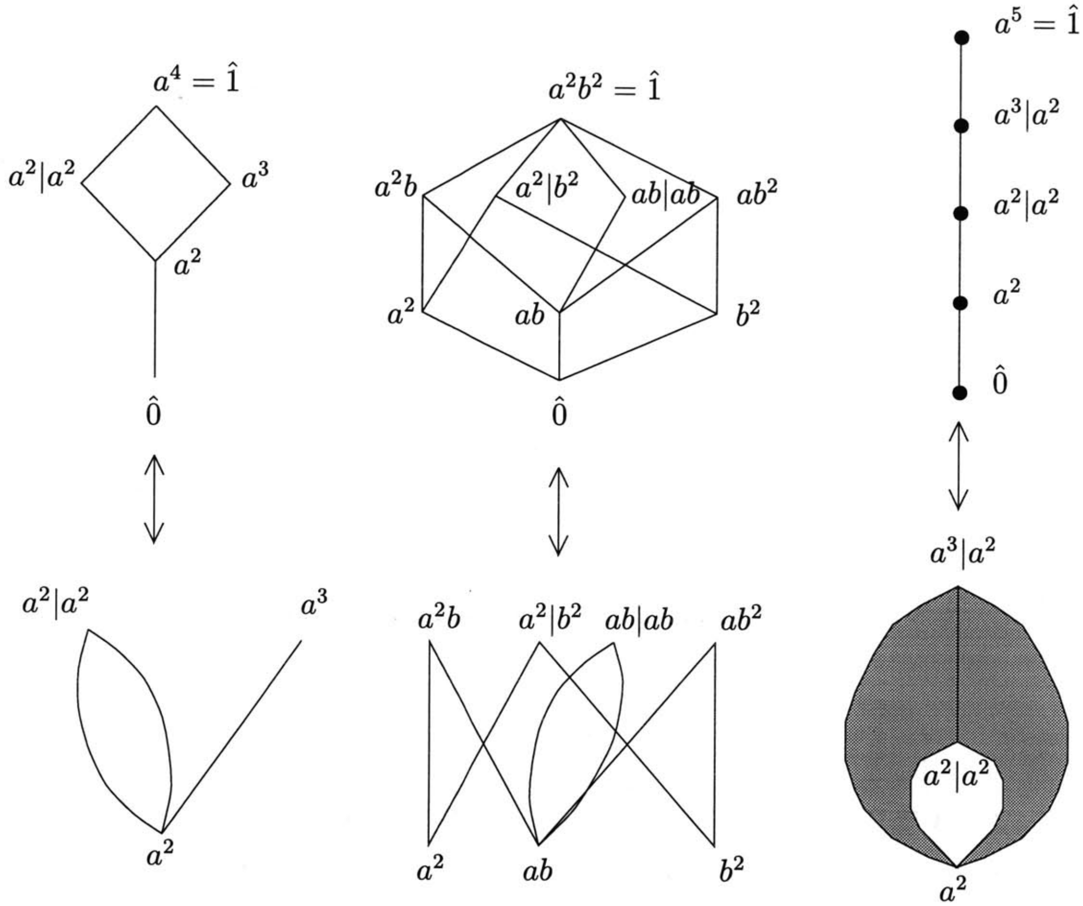

Figure 6-2: Multiset partition posets and their refinement complexes

Each i-refinement sequence gives rise to an i-cell in the **refinement complex**. Figure 6-2 gives examples of multiset partition posets together with their refinement complexes. Poset elements and refinement complex 0-cells are labelled with the non-trivial blocks of the corresponding multiset partition. In the case of the poset of partitions of $\{a^5\}$, Figure 6-2 depicts a subposet together with the corresponding subcomplex of the refinement complex. We suggest this example together with Figures 6-1 and 6-3 as examples of exactly how refinement sequences differ from chains. These examples are chosen to illustrate how a chain may give rise to more than one face, and how this may happen even when the chain skips ranks, as in the example of partitions of $\{a^5\}$.

Theorem 6.1.1 *The reduced Euler characteristic of the refinement complex for the poset of partitions of* $\{a_1^{n_1}, \ldots, a_l^{n_l}\}$ *is the deformed Möbius function* $\mu'(\hat{0}, a_1^{n_1} \ldots a_l^{n_l})$.

PROOF. Suppose this is true for all partitions of rank less than k and let $k = n_1 + \cdots + n_l$. For any intermediate element u of the poset of partitions of $\{a_1^{n_1}, \ldots, a_l^{n_l}\}$ we can assume that $\mu'(\hat{0}, u)$ equals the reduced Euler characteristic of our cell complex restricted to this interval. However, this is just the alternating sum of i-faces in our cell complex restricted to this interval. Hence $-\mu'(\hat{0}, u)$ is the alternating sum of i-faces in the entire cell complex which have u as their maximal element. Recall that $\mu'(\hat{0}, a_1^{n_1} \ldots a_l^{n_l}) = -\sum_{\hat{0} < u < a_1^{n_1} \ldots a_l^{n_l}} \mu'(\hat{0}, u)$. Hence, it is the sum over maximal elements of refinement sequences of the restricted Euler characteristics, so altogether it gives the Euler characteristic of the whole complex, as desired. $\square$

An alternate way to define a refinement complex is by starting with the order complex of a partition lattice and then gluing together certain vertices and faces as follows. Begin with the partition lattice Π_n obtained by an operation called polarization in [HRW]. One replaces a set of identical letters by a set with the letters distinguished by indices, so for example $\{b^n\}$ is replaced by $\{b_1, \ldots, b_n\}$. The identification then takes place on the order complex of Π_n. Two order complex faces are glued together in the refinement complex if the multiset partition poset elements giving rise to the vertices in both faces agree and if the two chains are refined in the same way.

6.2 The Euler characteristic of refinement complexes

This section gives combinatorial proofs of formulas for the reduced Euler characteristic of refinement complexes for two classes of multisets. We first consider the poset of partitions of $\{a^n\}$. In this case, the order complex is contractible because it is a cone; in contrast $\mu'_P(\hat{0}, a^n) = (-1)^{n-1}$ whenever n is a power of 2, and $\mu'_P(\hat{0}, a^n)$ equals 0 otherwise.

Proposition 6.2.1 *Let P be the poset of partitions of $\{a^n\}$. Then $\mu'_P(\hat{0},\hat{1}) = (-1)^{n-1}$ when n is a power of 2, and otherwise $\mu'_P(\hat{0},\hat{1}) = 0$.*

PROOF. Suppose this is true for rank less than n. Let $n_1+\cdots+n_k = n$. By induction, $\mu'_P(\hat{0}, a^{n_1}|\ldots|a^{n_k}) = 0$ for $k > 1$ unless each n_i is a power of 2. Let $m_1 = |\{n_i | n_i = 1\}|$. Induction implies that $\mu'_P(\hat{0}, a^{n_1}|\ldots|a^{n_k}) = (-1)^{k-m_1}$ if each $n_i = 2^j$ for some $j \geq 0$ and if $k > 1$. Hence, it suffices to restrict to partitions where each block has size a power of 2 and to give a bijection between such partitions of $\{a^n\}$ involving an even number of nontrivial parts and of those involving an odd number of nontrivial parts. There is such a correspondence which pairs a partition having a unique largest block with the partition which has this block split into two equal parts. This implies the result for $k = 1$, as needed. □

The Euler characteristic formula for the refinement complex of the poset of partitions of $\{a, b_1^{n_1}, \ldots, b_k^{n_k}\}$ generalizes the result for Π_n that $\mu(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$ because $\mu' = \mu$ in the case of the partition lattice Π_n.

Theorem 6.2.1 *If P is the poset of partitions of $\{a, b_1^{n_1}, \ldots, b_k^{n_k}\}$, and if $n - 1 = n_1 + \cdots + n_k$, then*

$$\mu'_P(\hat{0},\hat{1}) = (-1)^{n-1}\frac{(n-1)!}{\prod_{i=1}^k (n_i)!}.$$

PROOF. Suppose this is true for rank less than n. We decompose the set of partitions below the maximal element according to the content of the block containing the distinguished letter a. For any fixed block B such that $|B| < n-1$ and $a \in B$, note that

$$\sum_{\{u|B\in u\}} \mu'_P(\hat{0},\overline{u}) = 0$$

where $\overline{u}$ is obtained from u by deleting the block B. Since $\mu'_P(\hat{0},u) = \mu'(\hat{0},B)\mu'(\hat{0},\overline{u})$, we have

$$\sum_{\{u|B\in u\}} \mu'(\hat{0},u) = \mu'(\hat{0},B) \sum_{\{u|B\in u\}} \mu'(\hat{0},\overline{u}) = 0.$$

Therefore, we need only sum over all possible blocks B which contain a and satisfy $|B| = n-1$. For each such B, we sum over partitions u containing any particular such block B, yielding

$$\begin{aligned}
\sum_{B}\sum_{\{u|B\in u\}}\mu'(\hat{0},u) &= (-1)^{n-2}\sum_{j=1}^{k}\frac{(n-2)!}{(n_j-1)!\prod_{i\neq j}(n_i)!} \\
&= (-1)^{n-2}\sum_{j=1}^{k}\frac{(n-2)!(n_j)}{\prod_{i=1}^{k}(n_i)!} \\
&= (-1)^{n-2}\frac{(n-2)!}{\prod_{i=1}^{k}(n_i)!}\sum_{j=1}^{k}n_j \\
&= (-1)^{n-2}\frac{(n-1)!}{\prod_{i=1}^{k}(n_i)!}.
\end{aligned}$$

This implies

$$\mu'(\hat{0},\hat{1}) = -\sum_{B}\sum_{\{u|B\in u\}}\mu'(\hat{0},u) = (-1)^{n-1}\frac{(n-1)!}{\prod_{i=1}^{k}(n_i)!}.$$

□

6.3 Two generalized notions of lexicographic shellability

This section presents two generalized notions of lexicographic shellability, each of which implies that a complex is Cohen-Macaulay and that it has the homotopy type of a wedge of spheres concentrated in top dimension. These two versions of shellability are compatible, and we will employ both simultaneously in Section 6.4.

Let us first recall a few facts about traditional lexicographic shelling so as to point out the differences in what we do. In a poset lexicographic shelling, each intersection $F_j \cap \left(\cup_{i=1}^{j-1} F_i\right)$ is determined by the descents in the saturated chain C giving rise to F_j: for each descent there is a maximal face of codimension one in $F_j \cap \left(\cup_{i=1}^{j-1} F_i\right)$, because there is a lexicographically smaller saturated chain that agrees with C everywhere except at the descent. The fact that there is a unique increasing chain on each interval implies that all the maximal faces in $F_j \cap \left(\cup_{i=1}^{j-1} F_i\right)$ arise in this way, so $F_j \cap \left(\cup_{i=1}^{j-1} F_i\right)$ is pure of maximum possible dimension.

We will relax the requirement on a chain-labelling that every interval must have a unique increasing chain. We will only insist that every interval have a unique saturated chain which behaves topologically like an increasing chain.

Definition 6.3.1 *Let λ be a chain-labelling and let $C = \hat{0} \prec u_1 \prec \cdots \prec u_{i-2} \prec u_{i-1} \prec u_i = u$ be a saturated chain from $\hat{0}$ to u. The labelling λ gives rise to an* **honest ascent** *at u_{i-1} if $\lambda(u_{i-2}, u_{i-1}) \leq \lambda(u_{i-1}, u_i)$ and if there is no lexicographically smaller saturated chain from $\hat{0}$ to u which differs from C only at u_{i-1}. Any ascent which is not an honest ascent is a* **swap ascent**.

We analogously distinguish between descents based on how they will behave topologically in a lexicographic shelling.

Definition 6.3.2 *Let λ be a chain-labelling and let $C = \hat{0} \prec u_1 \prec \cdots \prec u_{i-2} \prec u_{i-1} \prec u_i = u$ be a saturated chain from $\hat{0}$ to u. The labelling λ gives rise to an* **honest descent** *at u_{i-1} if $\lambda(u_{i-2}, u_{i-1}) > \lambda(u_{i-1}, u_i)$ and if there is some lexicographically smaller saturated chain from $\hat{0}$ to u which differs from C only at u_{i-1}. Any descent which is not an honest descent is a* **swap descent**.

Thus, honest ascents and swap descents behave topologically like ascents would in a CL-labelling while honest descents and swap ascents both play the role of descents. This leads to a notion of lexicographic shellability where each interval is required to have a unique saturated chain consisting of only honest ascents and swap descents, and where these must be the lexicographically smallest chains on each interval. Our labelling of refinement complexes in Section 6.4 will involve honest ascents, swap ascents and honest descents, but swap descents will not come up at all.

Kozlov proved that a CC-labelling is the most general possible notion of poset lexicographic shellability and he gave several equivalent formulations [Ko]; it turns out that our definition amounts to the same thing as one of these formulations, phrased slightly differently. In Kozlov's motivating example, an application to the topology of intersection lattices of some subspace arrangements, he finds one of the other formulations more useful.

Theorem 6.3.1 *If a poset possesses a chain-labelling such that each interval has a unique saturated chain consisting entirely of honest ascents and swap descents, and if this is always the lexicographically smallest chain on the interval, then the labelling induces a lexicographic shelling.*

PROOF. The labelling induces an ordering $F_1, \dots, F_k$ on facets such that $F_j \cap \left(\cup_{i=1}^{j-1}\right)$ is pure of maximal possible dimension for each $2 \leq j \leq k$ by the same reasoning one applies to an ordering induced by a CL-labelling. This is discussed in more detail in [Ko]. □

The other direction in which we extend lexicographic shellability is motivated by the fact that refinement complexes are not always simplicial complexes.

Definition 6.3.3 *A regular cell complex K is* **quasi-simplicial** *if the closure of each j-cell is homeomorphic to the standard closed j-simplex by a homeomorphism which carries i-cells to i-simplices for all $i \leq j$.*

We will assume implicitly that all quasi-simplicial cell complexes are regular, and sometimes we will refer to i-cells as i-faces and 0-cells as vertices. We note that quasi-simplicial cell complexes may alternatively be defined in a very natural way using simplicial sets.

Definition 6.3.4 *A quasi-simplicial cell complex is* **pure** *if all the maximal cells have the same dimension.*

Definition 6.3.5 *A quasi-simplicial pure cell complex with maximal faces of dimension n is* **balanced** *if the vertices may be colored with $n+1$ colors in such a way that no two vertices in a face have the same color.*

Recall that the order complex of a poset is a pure balanced simplicial complex in which vertices are colored by poset rank. To give a lexicographic shelling of a pure balanced cell complex, we first need to make sense of such things as the notion of interval. There may be multiple faces in a refinement complex with the same set of vertices, so specifying two comparable elements does not give enough information to determine an interval in the sense that will most naturally allow shelling arguments to generalize.

Definition 6.3.6 *An* **interval** *in a quasi-simplicial pure balanced cell complex is given by an edge in the complex. If there is an edge E between two vertices which are colored i and j, respectively, for $i < j$, then the interval determined by E consists of all faces on vertices colored $i, \ldots, j$ which include E.*

Rooted intervals are defined similarly.

Definition 6.3.7 *A* **rooted interval** *is specified by a face F with vertices colored $1, \ldots, i, j$ for $i < j$. This rooted interval consists of all faces G comprised of vertices colored $1, \ldots, j$ such that $F \subseteq G$.*

This distinction of specifying a face rather than a collection of vertices to determine an interval is at the heart of why refinement complexes turn out to be shellable while order complexes of multiset partition posets are not even Cohen-Macaulay. Figure 6-3 depicts the image of two refinement sequences in a refinement complex. This may be contrasted with Figure 6-1. Figure 6-3 might lead one to believe that refinement complexes are not Cohen-Macaulay, but cell complexes such as the one in Figure 6-3 are not the link of any face and must be considered as part of a larger cell complex.

In simplicial complexes, one may conclude from the existence of a shelling that the complex has the homotopy type of a wedge of spheres concentrated in top dimension and that the complex is Cohen-Macaulay. We will verify these implications for quasi-simplicial pure balanced cell complexes by pointing out how a few difficulties that arise may be resolved. Let us first make precise what it means for a quasi-simplicial pure balanced cell complex to be lexicographically shellable. An EL-labelling may be

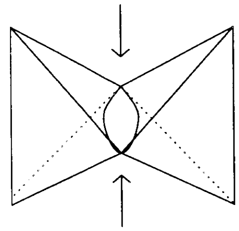

Figure 6-3: Two solid tetrahedra sharing two vertices

viewed as a labelling on edges in the order complex which are given by pairs of vertices colored $(i, i+1)$ for $1 \leq i \leq n-1$. In a CL-labeling, each edge label may depend on the choice of path through nodes colored consecutively from 1 to i. Labelling edges in the order complex rather than in the Hasse diagram allows us to generalize this to quasi-simplicial pure balanced cell complexes.

Definition 6.3.8 *A quasi-simplicial pure balanced cell complex is* **lexicographically shellable** *if there is a labelling such that each interval has a unique extension with increasing labels and if this is always the lexicographically smallest labelling on the interval.*

We believe this definition is justified by the following properties of such complexes. We will sometimes refer to maximal cells in a complex which we are shelling as facets to simplify notation in our arguments.

Proposition 6.3.1 *A lexicographic shelling of a quasi-simplicial balanced cell complex has the property that for any F_j, all the maximal cells in the intersection $F_j \cap (\cup_{i=1}^{j-1} F_i)$ have codimension one. This implies that the intersection of each maximal cell with the union of earlier ones is either the boundary of the cell or is contractible.*

PROOF. In a simplicial complex shelling, when the intersection of a facet with the union of earlier ones is a cone, this immediately implies that the intersection is contractible. This conclusion carries through to quasi-simplicial balanced cell complexes only because the intersection of a maximal cell with the union of earlier ones in the shelling order is actually a simplicial complex. Otherwise, the argument is virtually identical to the case of order complexes.

Let $F_1, F_2, \ldots, F_n$ be a lexicographic order on facets. Suppose $F_l \cap \left(\cup_{k=1}^{l-1} F_k\right)$ is not pure of dimension $dim(F_l - 1)$ for some l. Let G be a maximal cell in this intersection which has dimension less than $dim(F_l - 1)$. We claim that the vertices of G must be colored $1, \ldots, i-1, i, j, j+1, \ldots, n$ for some $i < j-1$. This is not hard to see, but we verify it below. The interval given by the edge containing the vertices colored i and j cannot consist entirely of honest ascents and swap descents, since there is a lexicographically smaller facet than F_l which also contains G. Hence, there is a codimension one face which strictly contains G and belongs to $F_l \cap \left(\cup_{k=1}^{l-1} F_k\right)$, a contradiction to G being maximal.

To check the claim, suppose the complement of the set of colors on vertices in G were not a single interval of the form $i+1, \ldots, j-1$. There would be multiple skipped intervals, the first of which has a lexicographically smaller extension than the extension agreeing with F_l, since otherwise G would not be maximal. The facet which only differs from F_l on this interval will also be lexicographically smaller than F_l and will strictly contain G, a contradiction. □

Corollary 6.3.1 *If a pure quasi-simplicial balanced cell complex is shellable, then it has the homotopy type of a wedge of spheres concentrated in top dimension.*

PROOF. The proof for shellable simplicial complexes applies directly since the first facet is contractible, and adding each subsequent facet either preserves the homotopy type or contributes a new sphere of top dimension. □

Corollary 6.3.2 *If a quasi-simplicial pure cell complex is shellable, then the underlying topological space is Cohen-Macaulay.*

PROOF. The first barycentric subdivision of a quasi-simplicial pure cell complex is a simplicial complex. Björner showed that the first barycentric subdivision of a shellable simplicial complex is shellable [Bj2, p.173], and his argument applies without need for modification to show that the first barycentric subdivision of a shellable quasi-simplicial balanced cell complex is also shellable. This implies that the barycentric subdivision is Cohen-Macaulay. A result of Munkres in [Mu2, p.117,121-123] implies that Cohen-Macaulayness does not depend on choice of triangulation of a topological space. Since taking a barycentric subdivision does not change the topological space, we are done. □

The topological definition of Cohen-Macaulayness for simplicial complexes does not carry over so well to cell complexes. For example, if a cell complex consists of two 1-cells whose closures share two 0-cells, then the link of either of these 0-cells should be a pair of 0-cells. We do not prove a cell complex analogue to the topological definition of Cohen-Macaulayness for refinement complexes directly because of examples such as this.

6.4 Shellability of refinement complexes

In this section, we prove that all refinement complexes are shellable. In the special case where one of the letters being partitioned has multiplicity one, namely that of partitions of $\{a, b_1^{n_1}, \ldots, b_k^{n_k}\}$, we give a homology basis of size $\frac{(n-1)!}{\prod_{i=1}^{k}(n_i)!}$ where $n-1 = n_1 + \cdots + n_k$. One reason to pay extra attention to this special case is that in this case any homology basis will have the same size as a basis for $Lie[a, b_1^{n_1}, \ldots, b_k^{n_k}]$, suggesting some possibility of an analogue to a relationship that exists between the partition lattice and $Lie[a_1, \ldots, a_n]$. (cf. Question 6.4.2 and see [Ba] for discussion of results in the partition lattice setting.)

We begin with another special case, that of partitions of $\{a^n\}$. The argument we give in this case contains the basic idea needed for shelling all refinement complexes. We work with the dual poset, since a lexicographic shelling of the dual poset suffices to get a shelling of a refinement complex. Thus, we will treat the unrefined block $\{a^n\}$ as the minimal element and let $u \prec v$ if v is obtained from u by refining one block into two. At each step, we draw a bar separating the two resulting blocks by the convention that we place the block with the smaller resulting word to the left of the bar. We label the refinement step with the consequent bar position. Note that bar position is well-defined at each step for two reasons:

1. The choice of label is allowed to depend on the choice of saturated refinement sequence below this step, which means that the sequence of earlier bar insertions imposes an order on the blocks.

2. Refinement sequences depend not only on what type of block is to be split, but on which actual block gets split, so that when a bar is to be placed in one of several identical blocks, the choice of block is specified.

It turns out that the lexicographic order on saturated refinement sequences induced by this labelling gives a shelling of the refinement complex, although this labelling is not in general a CL-labelling. Recall our plan of treating certain ascents which we call swap ascents as if they are descents. Suppose two consecutive steps consist of placing bars in position i followed by j for $i < j$, and suppose that i' is the position of the rightmost bar to the left of position i before a bar is inserted in position i. These two consecutive bar insertions at positions i and j comprise a swap ascent if $j - i < i - i'$. This pair of bar insertions behaves like a descent in that there is a lexicographically smaller refinement sequence differing only at this ascent. This comes from placing a bar in position $i' + j - i$ and then in position j, and otherwise leaving the refinement sequence unchanged. One may easily check that this lexicographically smaller refinement sequence indeed obeys the convention of placing the smaller word on the left at each refinement step. Furthermore, this is the only possible type of swap ascent in the refinement complex for $\{a^n\}$.

Theorem 6.4.1 *If P is the poset of partitions of $\{a^n\}$ with saturated chains ordered lexicographically as described above, then each interval has a unique refinement sequence consisting only of honest ascents (and swap descents). This is the lexicographically smallest saturated refinement sequence on the interval.*

Proof. First we construct a saturated refinement sequence consisting entirely of honest ascents for an arbitrary interval. Place bars left to right in the leftmost block to be refined so that the parts created in the block are nondecreasing in size when read left to right. Similarly refine the remaining blocks progressing left to right. By definition, such a refinement sequence is free of both descents and swap-ascents.

Next we show that there cannot be two different saturated refinement sequences on an interval both consisting entirely of honest ascents and swap descents. Note that every descent is an honest descent because whenever two consecutive steps consist of placing bars in position j and then i for $i < j$, one may reverse the order of insertion to obtain a legitimate saturated refinement sequence which is lexicographically smaller. Thus, we need only show there are not two saturated refinement sequences consisting entirely of honest ascents. This follows from the fact that in a refinement complex the choice of interval to be refined includes a choice of which blocks are split to form what types of new blocks. Since bars must be placed from left to right in the blocks, and the parts resulting from refinement within each block must be nondecreasing in size when read left to right, there is a unique way to do this.

Finally, any saturated refinement sequence which has an honest descent or a swap ascent cannot possibly be lexicographically smallest, so the saturated refinement sequence consisting entirely of honest ascents must itself be lexicographically smallest.
□

The proof for multiset partition posets in general will be quite similar, but is written somewhat more formally. Figure 6-1 gives an example of how the condition on intervals fails for order complexes of multiset partition posets. In the definition of interval in quasi-simplicial balanced cell complexes, we specify not only the minimal and maximal poset element on the interval, but also choose an edge between them. (See Figure 6-3 for an example.) One may check that neither the face colored $1, \dots, i$ nor the face colored $j, \dots, n$ in the intersection in Figure 6-3 is a maximal face in the intersection of a facet with the union of earlier ones in a shelling; thus, structures such as the one shown in Figure 6-3 do not preclude an intersection $F_j \cap \left(\cup_{i=1}^{j-1} F_i\right)$ from being pure.

Corollary 6.4.1 *Refinement complexes of $\{a^n\}$ are shellable.*

Proof. Apply Corollary 6.3.1, Corollary 6.3.2, Theorem 6.4.1 and the fact that a CC-labelling leads to a shelling. □

Theorem 6.4.2 *The refinement complex of $\{a^n\}$ has the homotopy type of a sphere of dimension $n-2$ if n is a power of 2, and it is contractible otherwise.*

Proof. Keeping with the analogy to a CL-labelling, we need only count refinement sequences that have no honest ascents. If n is a power of 2, then the facet obtained by greedily placing bars as far to the right as possible attaches along its entire boundary. We first split a^n into two blocks of equal size. By induction, we may then greedily refine first the right block and then the left block, each into components of size one. We call this the greedy facet. Each ascent in the greedy facet comes from placing a bar in the middle of one block and then in the middle of the resulting right half, so the ascents are all swap ascents.

We claim that a facet will contain an honest ascent if it ever deviates from the greedy facet. By induction, the first bar must split a^n into two blocks each of size equalling a power of 2, since any refinement sequence which is free of honest ascents must next completely refine the right block and then completely refine the left one. Furthermore, the first two bar insertions must comprise an ascent, so it must be a swap ascent. If the left block created at the first step is smaller than the right block, then the left block is at most half the size of the right block (since each has size a power of 2), but then a swap ascent would not be possible. Hence, the first bar must divide a^n into two equal parts. By induction, the refinement of the right half must agree with the greedy facet, and then the subsequent refinement of the left half must also necessarily agree with the greedy facet. □

It might be tempting to reorder the labels so that swap-ascents become descents to make our labelling into a CL-labelling. However, such an order relation could not possibly be transitive. This suggests the following question.

Question 6.4.1 *Is there some way of labelling saturated refinement sequences so that every interval has a unique increasing chain, in the sense of a CL-labelling?*

Let us now give a variant of our labelling by bar position, because this will be useful for more general multiset partition posets and will also allow us to count cycles in a homology basis in some interesting cases. Label refinement steps with ordered pairs (i, S) where i is the number of bars to the left of the bar to be inserted and S is the block immediately to the left of the new bar, namely the multiset between the new bar and the bar to its left. We denote the word of content S with letters arranged in increasing order by w_S. Let $(i, S) > (i', S')$ whenever 1) $i > i'$ or 2) $i = i'$ and $|S| > |S'|$. If $i = i'$ and $|S| = |S'|$, then we let $(i, S) < (i', S')$ if w_S is lexicographically smaller than $w_{S'}$. We will show that for all multiset partition posets the induced lexicographic order on refinement sequences gives a shelling.

In fact, any ordering on multisets satisfying the condition

$$w \leq w' \Rightarrow w \leq ww',$$

which we call the **lengthening condition (LC)**, will lead to a shelling. In particular, another order on words will lead to a nice homology basis in some cases. Let us denote by $\preceq$ any ordering on words satisfying (LC).

Our plan is to show that property (LC) implies the next two lemmas which together yield lexicographic shellability. In particular, we show that refinement complexes are CC-shellable quasi-simplicial pure balanced cell complexes. In the next two lemmas, let $B_{i,j}$ be the jth block encountered reading left to right in the multiset partition of rank i in a saturated refinement sequence, and let $k(i)$ denote the number of bars to the left of the bar which is inserted at step i in this saturated refinement sequence.

The first lemma checks that the lexicographically smallest saturated chain on each interval consists entirely of honest ascents.

Lemma 6.4.1 *In a saturated refinement sequence $\pi_0, \dots, \pi_n$, if two consecutive labels $(k(i), B_{i,k(i)})$ and $(k(i+1), B_{i+1,k(i+1)})$ do not comprise an honest ascent, then there* *exists another refinement sequence* $\pi_0, \ldots, \pi_{i-1}, \pi'_i, \pi_{i+1}, \ldots, \pi_n$ *whose label sequence is lexicographically smaller.*

PROOF. There are three cases to consider: a descent of the form $[(k, S), (k, S')]$, a descent $[(k, S), (j, S')]$ for $j < k$ and a swap ascent, namely an ascent of the form $[(k, S), (k + 1, S')]$ where $S' \preceq S$. These are shown in Figure 6-4. In each case, we show a lexicographically smaller refinement sequence immediately to the right of the refinement sequence containing a descent or swap ascent. We need to make sure that each of these lexicographically smaller refinement sequences is valid in the sense that every time a block is subdivided, the smaller word is placed to the left of the bar. In most cases this is clear; otherwise, we use the fact that $\preceq$ satisfies the lengthening condition. For example, to verify that $T \preceq S \cup U$ in Figure 6-4(c), we begin by noting that $T \preceq S$ (since this is a swap ascent), and $T \preceq U$ (since otherwise the bottom picture on the left would be $S|U|T$). Now either $S \preceq U$ or $U \preceq S$, so by the lengthening condition either $S \preceq S \cup U$ or $U \preceq S \cup U$. In either case, it follows by transitivity that $T \preceq S \cup U$. The other verifications are similar. $\square$

Lemma 6.4.2 *For any refinement sequence* $\pi_0, \ldots, \pi_i, \pi'$*, in which* $\pi_0, \ldots, \pi_i$ *is a saturated refinement sequence, there is a unique way of extending this to a saturated refinement sequence* $\pi_0, \ldots, \pi_i, \pi_{i+1}, \ldots, \pi_j = \pi'$ *consisting entirely of honest ascents.*

PROOF. Let us describe how to refine π_i to π_{i+1}; the rest of the refinement sequence follows by induction. Among the blocks $B_{i,0}, \ldots, B_{i,m}$ of π_i, find the lowest-numbered one (namely the block as far to the left as possible) which gets subdivided into smaller pieces in π'. Let $B_{i,k}$ be this block. Assuming that $B_{i,k}$ is broken into pieces $C_0, \ldots, C_r$ in π' with these numbered so that $C_0 \preceq \ldots \preceq C_r$, we define π_{i+1} to be the partition obtained from π_i by breaking $B_{i,k}$ into blocks C_0 and $\cup_{i=1}^{r} C_i$.

Note that the label of this refinement step is (k, C_0) and that this label is lexicographically smallest among the labels at stage $i + 1$ in saturated chains extending $\pi_0, \ldots, \pi_i, \pi'$. By induction, it follows that our construction indeed gives the lexicographically smallest saturated extension of $\pi_0, \ldots, \pi_i, \pi'$. Lemma 6.4.1 then implies

$$
\begin{tikzcd}
S|T|\cdots|U|V \arrow[d, dash, "{(k(i+1),S)}"] \\
S\cup T|\cdots|U|V \arrow[d, dash, "{(k(i),U)}"] \\
S\cup T|\cdots|U\cup V
\end{tikzcd}
\quad \text{vs.} \quad
\begin{tikzcd}
S|T|\cdots|U|V \arrow[d, dash, "{(k(i),U)}"] \\
S|T|\cdots|U\cup V \arrow[d, dash, "{(k(i+1),S)}"] \\
S\cup T|\cdots|U\cup V
\end{tikzcd}
$$

(a) Descent — $k(i) > k(i+1)$

$$
\begin{tikzcd}
S|T|U \arrow[d, dash, "{(k(i),S)}"] \\
S\cup T|U \arrow[d, dash, "{(k(i),S\cup T)}"] \\
S\cup T\cup U
\end{tikzcd}
\quad \text{vs.} \quad
\begin{tikzcd}
S|T|U \arrow[d, dash, "{(k(i+1),T)}"] \\
S|T\cup U \arrow[d, dash, "{(k(i),S)}"] \\
S\cup T\cup U
\end{tikzcd}
\qquad
\begin{tikzcd}
S|T|U \arrow[d, dash, "{(k(i+1),T)}"] \\
S|T\cup U \arrow[d, dash, "{(k(i),S)}"] \\
S\cup T\cup U
\end{tikzcd}
\quad \text{vs.} \quad
\begin{tikzcd}
S|T|U \arrow[d, dash, "{(k(i+1),S)}"] \\
T|S\cup U \arrow[d, dash, "{(k(i),T)}"] \\
S\cup T\cup U
\end{tikzcd}
$$

(b) Descent — $k(i) = k(i+1)$

(c) Swap ascent — $T \preceq S$

Figure 6-4: Descents and swap ascents in refinement sequences

that the resulting saturated refinement sequence must consist entirely of honest ascents.

It remains to show that no other saturated extensions of $\pi_0, \ldots, \pi_i, \pi'$ also consist entirely of honest ascents. To avoid both descents and swap ascents, bars must be inserted left to right and each component in π_i must be broken into blocks which are nondecreasing (under our partial order $\preceq$ on multisets) when read left to right. Just as in the case of partitions of $\{a^n\}$, our definition of interval for quasi-simplicial cell complexes implies that there is a unique such saturated extension on each interval. $\square$

Applying results of section 6.3 validating generalized notions of lexicographic shellability, we may conclude the following.

Theorem 6.4.3 *Every refinement complex is shellable.*

PROOF. This is immediate from the above lemmas since we have shown that every interval in a refinement complex has a unique saturated refinement sequence consisting entirely of honest ascents and swap descents, and that this is always lexicographically smallest. □

Corollary 6.4.2 *The underlying topological space of a refinement complex is Cohen-Macaulay.*

Counting the analogue of decreasing chains yields the following.

Corollary 6.4.3 *The homotopy type of the refinement complex for the poset of partitions of* $\{a, b_1^{n_1}, \ldots, b_k^{n_k}\}$ *is a wedge of*

$$\frac{(n-1)!}{\prod_{i=1}^{k}(n_i)!}$$

spheres each of dimension $n-2$ *where* $n_1 + \cdots + n_k = n-1$.

Proof. Theorem 6.4.3 implies the homotopy type is a wedge of spheres concentrated in top dimension, and our Euler characteristic formula then counts these spheres.

Alternatively, we give a homology basis with cycles which we may easily count. Let us order words by $S \preceq S'$ if $a \in S$ and $a \notin S'$, otherwise using the previously given order. That is, otherwise let $S \preceq S'$ if $|S| < |S'|$ or if $|S| = |S'|$ and w_S is lexicographically smaller than $w_{S'}$. This satisfies the condition (LC), and it is clear that the chains attaching along their entire boundary are in correspondence with the rearrangements of $ab_1^{n_1} \ldots b_k^{n_k}$ which begin with the distinguished letter a. Each such rearrangement gives rise to a saturated refinement sequence consisting entirely of descents. These are the only refinement sequences attaching along their entire boundary because if a bar is ever placed anywhere but in the rightmost possible position, then there must be an honest ascent. □

Corollary 6.4.4 *The refinement complex for the poset of partitions of* $\{a, b_1^{n_1}, \ldots, b_k^{n_k}\}$ *has a homology basis indexed by the distinct ways of ordering the letters with the unique copy of the letter* a *appearing first give rise to a homology basis.*

Question 6.4.2 *When a multiset has a distinguished letter, is there a nice correspondence between elements of some homology basis and generators of a corresponding subspace of a free Lie algebra? Notice that the Lyndon words are exactly those words starting with the distinguished letter, if we make this letter lexicographically smallest. Since the Lyndon words formed from* $\{a, b_1^{n_1}, \ldots, b_k^{n_k}\}$ *comprise a basis for* $Lie[ab_1^{n_1} \ldots b_k^{n_k}]$*, the two bases have the same size. Do results of Barcelo, Garsia, Hanlon, Stanley, et al. regarding set partitions, free Lie algebras and their representations generalize nicely to this setting for some suitable choice of homology basis? (cf. [Ba])*

Let m_i be the number of letters occuring with multiplicity i in $\{a_1^{n_1}, \ldots, a_k^{n_k}\}$. The group $S_{m_1} \times S_{m_2} \times \ldots$ acts on the partitions of multisets of $\{a_1^{n_1}, \ldots, a_k^{n_k}\}$ by letting S_{m_i} permute the values of the letters which occur with multiplicity i.

Question 6.4.3 *Is the induced action on the homology of the order complex or the refinement complex related in a nice way to a representation of* $Lie[a_1^{n_1} \cdots a_k^{n_k}]$*?*

Let $T(n_1, \ldots, n_k)$ be the set of words of content $a_1^{n_1} \cdots a_k^{n_k}$ which are consistent with saturated chains attaching along their entire boundary. That is, in choosing a saturated refinement sequence we have a convention for ordering blocks split by bars at each successive step. Each refinement sequence gives rise to a word with the letters ordered in the same way the blocks of size one are ordered at the conclusion of the refinement. We let $T(n_1, \ldots, n_k)$ be the set of words which result in this fashion from saturated refinement sequences which attach along their entire boundary.

Corollary 6.4.5 *The function* μ' *counts cycles in top homology, so* $\mu'(\hat{0}, a_1^{n_1} \ldots a_k^{n_k}) = |T(n_1, \ldots, n_k)|$ *in general.*

It would of course be nice to have a better description of $T(n_1, \ldots, n_k)$.

Question 6.4.4 *Is there a closed formula for* $|T(n_1, \ldots, n_k)|$*? Is there at least a more explicitly described collection of combinatorial objects indexed by the elements of* $T(n_1, \ldots, n_k)$.

We compile these and other open questions in Chapter 7.

Chapter 7

Open questions

We have included open questions throughout. For convenience, this chapter collects these questions together along with several new ones. These presumably vary quite a bit in difficulty. We are actively investigating a few of these questions, while we have not considered many of them at all beyong formulating the questions.

1. Are there other well-known classes of posets which are amenable to chain decompositions leading to flag f-vector formulas in the spirit of Chapter 3? Recall that F_P is a symmetric function whenever P is locally rank-symmetric. We believe that geometric lattices for which F_P is a symmetric function may be particularly good candidates because they have EL-labellings which may interact well with Lemma 3.1.1. In addition, posets which may be embedded into geometric lattices in a nice way (e.g. products of chains) may also be good candidates.

2. Can one determine F_P for the lattice of subspaces of a finite-dimensional vector space over a finite field by giving a chain decomposition? Perhaps this would come from partially ordering the boolean sublattices given by the different choices of basis.

3. Is there a better formula for F_P for graded monoid posets in special cases? Is there some choice of monomial term order which gives a particularly nice formula for F_P, at least in special cases?

4. Can one obtain interesting new symmetric function identities (or interesting new proofs) by computing F_P in two different ways for some poset (or some collection of chains in a poset)? Is there any meaningful connection to plethysm?

5. Generalize results of Simion and Stanley about the monoid of multiplicative functions, as discussed in the last section of [SS, p. 25-32]. In particular, their work might be generalized to several variables such as the two alphabets u and v which arise in Theorem 4.1.1 in the study of shuffle posets of multisets, or more generally the k alphabets in Theorem 4.2.2.

6. Characterize all possible orbits of local symmetric group algebra actions on posets (or on lattices). Thus, allow adjacent transpositions to take maximal chains to linear combinations of maximal chains each of which only differ from the original chain at the rank specified by the adjacent transposition being applied. One might also consider Hecke algebra local actions.

7. Is there a nice interpretation for the symmetric group representation on maximal chains with Frobenius characteristic F_P or ωF_P for shuffle posets of multisets? Is there a local group algebra action on the maximal chains in shuffle posets of multisets with Frobenius characteristic F_P or ωF_P

8. Characterize local action orbits for the action of the affine symmetric group or for other Coxeter groups. We believe this should be possible with a tiling analysis similar to our argument for local symmetric group actions on lattices, at least in the case of simply-laced Coxeter groups.

   We suggest the following definition for a local action of a Coxeter group W. Let $w_{s_1} \cdots w_{s_k}$ and $w_{t_1} \cdots w_{t_l}$ be reduced expressions in W, let W act on a set S, and let $p \in S$. Let $\overline{w_{a_1} \cdots w_{a_m}(p)}$ be the element of W obtained from $w_{a_1} \cdots w_{a_m}$ by deleting each simple reflection w_{a_i} which acts trivially on $w_{a_{i+1}} \cdots w_{a_m}(p)$. We then say that W acts locally on S if $w_{s_1} \cdots w_{s_k} p = w_{t_1} \cdots w_{t_l} p$ implies $\overline{w_{s_1} \cdots w_{s_k}(p)} = \overline{w_{t_1} \cdots w_{t_l}(p)}$ no matter which element of S and which reduced expressions we choose.

9. In multiset partition posets, what is $\mu'(u,v)$ for arbitrary $u \le v$? What about $\mu'(\hat{0},u)$ for general u? (Chapter 6 addresses the special cases of $\mu'(\hat{0},u)$ for $u = a^n$ and $u = ab_1^{n_1}\cdots b_k^{n_k}$.)

10. Are there nice formulas for the functions μ' on multiset intersection posets?

11. Is there a labelling of saturated refinement sequences that is the balanced cell complex analogue of a CL-labelling rather than a CC-labelling?

12. Is there a nice characterization of a homology basis for the refinement complex of the poset of partitions of $\{a_1^{n_1},\ldots,a_k^{n_k}\}$ when none of the n_i equal one? Is there an explicit formula for $\mu'(\hat{0},a_1^{n_1}\cdots a_k^{n_k})$ in general?

13. Is the order complex of the poset of partitions of $\{a_1^{n_1},\ldots,a_k^{n_k}\}$ shellable whenever the order complex of the poset of partitions of $\{a^{\max_{1\le i\le k} n_i}\}$ is shellable?

14. Is the order complex of a multiset partition poset homotopy equivalent to a wedge of spheres concentrated in top dimension? Note that the order complex of the poset of partitions of $\{a^n\}$ is contractible because it is a cone in which every maximal face includes the vertex given by the partition with one nontrivial block a^2. Is the topology of the order complex closely related to that of the corresponding refinement complex?

15. Is there a nice correspondence between elements of some homology basis for the refinement complex of $\{a,b_1^{n_1},\ldots,b_k^{n_k}\}$ and generators of $Lie[a,b_1^{n_1},\ldots,b_k^{n_k}]$? Notice that the Lyndon words are exactly those words starting with the distinguished letter, if we make this letter lexicographically smallest. Since the Lyndon words of content $ab_1^{n_1}\cdots b_k^{n_k}$ form a basis for $Lie[a,b_1^{n_1},\ldots,b_k^{n_k}]$, the two bases have the same size. Do results of Barcelo, Garsia, Hanlon, Stanley, et al. regarding set partitions, free Lie algebras and their symmetric group representations generalize to this setting? (cf. [Ba] for details and related references.)

16. If we let $m_i = |\{j|n_j = i\}|$ and $n = \sum_{i=1}^k n_i$, then the (finite) Young subgroup $S_{m_1}\times S_{m_2}\times\ldots$ of S_n acts on the partitions of the multiset $\{a_1^{n_1},\ldots,a_k^{n_k}\}$ by let-

ting S_{m_i} permute the values of the letters which occur with multiplicity i. Is the induced representation on the homology of the order complex or the refinement complex related in a nice way to a representation on $Lie[a_1^{n_1}, \ldots, a_k^{n_k}]$? We add as a word of caution that the homology bases given in Chapter 6 for refinement complexes of multiset partition posets are quite different than the homology basis generally studied in conjunction with symmetric group representations on the homology of the partition lattice.

17. Let $T(n_1, \ldots, n_k)$ be the set of words of content $a_1^{n_1} \cdots a_k^{n_k}$ which are consistent with saturated chains attaching along their entire boundary. That is, in choosing a saturated refinement sequence we have a convention for ordering blocks split by bars at each successive step. Each refinement sequence gives rise to a word with the letters ordered in the same way the blocks of size one are ordered at the conclusion of a refinement. We let $T(n_1, \ldots, n_k)$ count the words which result in this fashion from saturated refinement sequences attaching along their entire boundary. Is there a closed formula for $|T(n_1, \ldots, n_k)|$? Is there at least a more explicitly described collection of combinatorial objects indexed by the elements of $T(n_1, \ldots, n_k)$.

Appendix A

Background material

We review definitions and properties related to partially ordered sets, topological combinatorics, quasi-symmetric functions and symmetric functions. This is not meant to be comprehensive, but rather a collection of material we will assume. First we recall a few definitions that do not fall into any of these four categories.

Definition A.0.1 *A* **composition** *of n, denoted by α, is an ordered collection of positive integers $\alpha_1, \ldots, \alpha_k$ such that $\alpha_1 + \cdots + \alpha_k = n$.*

Definition A.0.2 *A* **multiset** *is a set in which items may occur with repetition.*

Definition A.0.3 *A basis B for an ideal I is a* **Gröbner basis** *with respect to a particular term order if the ideal $\langle LT(B) \rangle$ generated by the leading terms of B consists of the same monomials as the collection $LT(I)$ of all leading terms in I.*

A Gröbner basis for an ideal I gives a way of reducing each element in a ring R/I to the unique minimal element in its equivalence class. This provides a finite algorithm for determining whether any two elements of R are equivalent in R/I, so in some sense a Gröbner basis gives a way of dividing a polynomial by an ideal.

A.1 Partially ordered sets

This section is based on [St2].

Definition A.1.1 *Any relation $\leq$ satisfying the following three properties is a* **partial order**, *and a partially ordered set is often called a* **poset**.

1. *reflexivity: $x \leq x$*

2. *transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$.*

3. *anti-symmetry: $x \leq y$ and $y \leq x$ implies $x = y$.*

A poset has a **covering relation** $u \prec v$ if $u \leq v$ and $u \leq w \leq v$ implies $w = u$ or $w = v$. An **atom** is any poset element which covers $\hat{0}$. A **chain** is any collection of poset elements which are all comparable. When a poset has a minimal and maximal element, we denote these by $\hat{0}$ and $\hat{1}$, respectively. A poset with $\hat{0}$ and $\hat{1}$ is **ranked** if the maximal chains of all have the same length. In this case, we have a rank function ρ defined by $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ for $x \prec y$. We will only study finite, ranked posets.

A finite poset is represented by a graph called a **Hasse diagram**, which consists of a vertex for each poset element and an edge connecting any pair of elements u, v such that $u \prec v$. By convention, v is drawn above u whenever $u \prec v$. It is implicit that $u \leq v$ if there is some chain of covering relations $u = u_1 \prec \cdots \prec u_k = v$.

Definition A.1.2 *A finite* **lattice** *is a poset in which every pair of elements have a unique least upper bound and greatest lower bound. The least upper bound of u and v is called the* **meet** *of u and v and is denoted by $u \vee v$, while the greatest lower bound is called the* **join** *of u and v and is denoted by $u \wedge v$.*

Definition A.1.3 *A finite lattice L is* **geometric** *if every element is a join of atoms and if $\rho(x) + \rho(y) \geq \rho(x \vee y) + \rho(x \wedge y)$ for all $x, y \in L$.*

Definition A.1.4 *The* **intersection lattice** *of a subspace arrangement is the lattice of subspaces partially ordered by reverse inclusion with a minimal element adjoined to represent the entire arrangement and a maximal element representing the empty set.*

Definition A.1.5 *A lattice is* **distributive** *if the meet and join satisfy the distibutivity laws $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

Definition A.1.6 *An* **M-chain** *of a lattice L is a saturated chain C such that the sublattice generated by C and other other chain in L is distributive. Any lattice having an M-chain is* **supersolvable**.

Supersolvability implies EL-shellability which in turn implies Cohen-Macaulayness, two topological properties to be discussed in the next section.

Definition A.1.7 *A finite poset of rank n is* **rank-symmetric** *if the number of elements of rank i equals the number of elements of rank $n-i$ for $0 \leq i \leq n$. A finite, ranked poset is* **locally rank-symmetric** *if each interval is rank-symmetric.*

Stanley noticed that F_P is a symmetric function whenever P is locally rank-symmetric, because taking the dual of an interval in a locally rank-symmetric poset does not affect the rank generating function, and consequently F_P is fixed by adjacent transpositions. When F_P is a symmetric function, we say that P is **flag-symmetric**.

Definition A.1.8 *A* **symmetric chain decomposition** *of a finite, ranked poset is a decomposition of the poset elements into symmetrically placed saturated chains, by which we mean that the rank of the minimal element of such a saturated chain plus the rank of its maximal element must equal the rank of the poset.*

An important class of posets with symmetric chain decompositions are the products of chains, as discussed (for example) in [GK].

A.2 Topological combinatorics

Definition A.2.1 *The* **order complex** *of a finite poset P which has a minimal and maximal element is the simplicial complex with an $(i-1)$-face for each i-chain $\hat{0} < x_1 < \cdots < x_i < \hat{1}$ of comparable poset elements.*

Definition A.2.2 *The Möbius function μ of a poset is defined on intervals by*

1. $\mu_P(x, x) = 1$

2. $\mu_P(x,z) = -\sum_{x \le y < z} \mu(x,y)$ *for* $x \le z$ *in* P

The Möbius function $\mu(\hat{0},\hat{1})$ is also the reduced Euler characteristic of the order complex of P. Next we give two equivalent definitions for a shelling of a pure simplicial complex. Recall that a simplicial complex is **pure** if all its maximal faces have the same dimension.

Definition A.2.3 *Let $F_1,\ldots,F_m$ be a total order on the maximal faces of a pure simplicial complex, and let n be the dimension of each maximal face. This is a* **shelling** *order if $F_j \cap (\cup_{1\le i\le j-1} F_i)$ is a pure simplicial complex of dimension $n-1$ for each $2 \le j \le m$.*

Equivalently, an ordering $F_1,\ldots,F_m$ of facets is a shelling order if the collection of new faces introduced at each shelling step includes a minimal new face. Let us make this more precise. If a simplicial complex is viewed as a collection of sets, then the set of new faces at step j is the set of sets which belong to $\cup_{i=1}^{j} F_i$, but do not belong to $\cup_{i=1}^{j-1} F_i$. At each step, we require there to be a new set that is contained in all the other new sets.

For pure simplicial complexes, shellability implies both Cohen-Macaulayness and that the complex is homotopy equivalent to a wedge of spheres concentrated in top dimension. This follows from the fact that each shelling step either preserves the homotopy type or completes a new sphere of top dimension, depending on whether the intersection of the new facet with the union of previous facets is contractible or the new facet attaches along its entire boundary. Björner and Wachs have extended the notion of shellability to non-pure complexes in [BW2] and [BW3].

Definition A.2.4 *The* **link** *of a face F is the collection of faces G such that $G \cup F$ is also a face, but $G \cap F = \emptyset$; in defining link, we again treat a simplicial complex as a collection of sets. A simplicial complex is* **Cohen-Macaulay** *if for each face F, the reduced homology group $\tilde{H}_i(\mathrm{lk(F)})$ vanishes for $i < dim(\mathrm{lk(F)})$. A poset is* **Cohen-Macaulay** *if its order complex is Cohen-Macaulay.*

Reisner proved that the Stanley-Reisner ring of a Cohen-Macaulay simplicial complex is a Cohen-Macaulay ring, so this is one reason for interest in this property.

Definition A.2.5 *An edge-labelling λ of a finite, ranked poset is an* **EL-labelling** *if it satisfies the following two properties.*

1. *For each $u \le v$ there is a unique saturated chain $u = u_1 \prec \cdots \prec u_k = v$ such that $\lambda(u_1, u_2) \le \cdots \le \lambda(u_{k-1}, u_k)$.*

2. *Given any other saturated chain from u to v, the word given by its sequence of edge labels is lexicographically larger than the word $\lambda(u_1, u_2) \cdots \lambda(u_{k-1}, u_k)$.*

A related notion is that of a **CL-labelling**. Instead of the edge label $\lambda(u, v)$ only depending on u and v, it may depend on the choice of saturated chain from $\hat{0}$ to u. If a poset has an EL-labelling or a CL-labelling, then its order complex is shellable. In either case, any linear extension of the lexicographic order on label sequences induces a shelling, and we say that the poset has a **lexicographic shelling**; furthermore, the saturated chains with strictly decreasing labels index the cycles in a homology basis.

A.3 Quasi-symmetric functions and the flag f-vector

Definition A.3.1 *A power series $q(x)$ is* **quasi-symmetric** *if the coefficient of $x_{a_1}^{k_1} \ldots x_{a_n}^{k_n}$ in $q(x)$ equals the coefficient of $x_{b_1}^{k_1} \ldots x_{b_n}^{k_n}$ for any $a_1 < \cdots < a_n$ and any $b_1 < \cdots < b_n$, together with any choice of $k_1, \ldots, k_n \in \mathbb{N}$.*

In [Ge], Gessel defined a quasi-symmetric analogue of the monomial symmetric functions by

$$M_{(r_1, r_2 - r_1, \ldots, n - r_k)} = \sum_{i_1 < \cdots < i_{k+1}} x_{i_1}^{r_1} x_{i_2}^{r_2 - r_1} \ldots x_{i_{k+1}}^{n - r_k}.$$

These functions form a basis for the quasi-symmetric functions. In [Eh], Ehrenborg introduced an encoding, denoted F_P, for the flag f-vector of a poset in terms of the Hopf algebra of quasi-symmetric functions which was defined by Gessel in [Ge].

Recall that the **flag f-vector** of a finite poset of rank n is a function on subsets of $\{1,\ldots,n-1\}$. For each collection $S=\{r_1,\ldots,r_k\}\subseteq\{1,\ldots,n-1\}$, the evaluation of the flag f-vector, denoted by $\alpha_P(S)$, counts the chains passing exactly through ranks $r_1,\ldots,r_k$. Summing over multichains, the quasi-symmetric function F_P is defined for nontrivial, ranked posets as

$$F_P=\sum_{\hat{0}=t_0\leq t_1\leq\cdots\leq t_{k-1}<t_k=\hat{1}} x_1^{\rho(t_0,t_1)}x_2^{\rho(t_1,t_2)}\ldots x_k^{\rho(t_{k-1},t_k)},$$

where $\rho(x,y)$ is the difference in ranks of x and y. F_P encodes the flag f-vector in a natural way, namely, $F_P=\sum_{S\subseteq\{1,\ldots,n-1\}}\alpha_P(S)M_{(r_1,r_2-r_1,\ldots,r_k-r_{k-1},n-r_k)}$ for $S=\{r_1<r_2<\cdots<r_k\}$ and P a finite poset of rank n.

The **flag h-vector** is related to the flag f-vector according to the formula $\beta_P(S)=\sum_{T\subseteq S}(-1)^{|S-T|}\alpha_P(T)$ where $\beta_P(S)$ is the coefficient indexed by S in the flag h-vector. This is often a more convenient way of expressing the same data. The poset in Figure 1-2 has flag f-vector $(1,1,2,2)$ and flag h-vector $(1,0,1,0)$ if the coordinates are indexed by $\emptyset,\{1\},\{2\}$ and $\{1,2\}$, respectively.

When F_P is a symmetric function, then the poset P is called **flag-symmetric**. In this case, the coefficient of $M_{(s_1,\ldots,s_k)}$ in the expression for F_P does not depend on the order of the elements $s_1,\ldots,s_k$ where $(s_1,\ldots,s_k)$ is a composition of n; monomial quasi-symmetric functions may be grouped into monomial symmetric functions, yielding $F_P=\sum_{\lambda\vdash n}\alpha_P(\lambda_1,\lambda_1+\lambda_2,\ldots,\lambda_1+\cdots+\lambda_k)m_\lambda$ for a finite poset of rank n, summing over partitions rather than compositions of n.

Gessel originally provided another basis for the quasi-symmetric functions, with elements which we denote by $L_{S,n}$, defined as

$$L_{S,n}(x)=\sum_{\substack{a_1\leq\cdots\leq a_n\\ a_i<a_{i+1}\ \text{if}\, i\in S}} x_{a_1}\ldots x_{a_n}.$$

This basis is related to the monomial basis of quasi-symmetric functions by $L_{S,n}=\sum_{T\subseteq S}M_T$, and it follows that $F_P=\sum\beta_P(S)L_{S,n}(x)$ for P of rank n, as is shown in [St5]. When a poset P is Cohen-Macaulay, then the coefficients $\beta_P(S)$ are nonnegative, since $\beta_P(S)$ counts the dimension of the homology group of the order complex of P restricted to ranks belonging to S, as discussed in [St2]. It follows from results of Gessel in [Ge] that the coefficients $\beta_P(S)$ are also nonnegative whenever F_P is Schur-positive.

A.4 Symmetric functions

This section is based primarily on Sagan [Sa].

Definition A.4.1 *A* **symmetric function** *is a function f in $k[x_1, x_2, \ldots]$ which is an (infinite) sum of monomials and which satisfies $\sigma f = f$ for every $\sigma \in S_\infty$ permuting the indices of the variables.*

It suffices to restrict to the finite collection of monomials involving only $x_1, \ldots, x_n$ to study representations of S_n or check identities involving symmetric functions of degree at most n.

Let us recall the definitions of several different symmetric functions bases that we will need. These will be indexed by partition, so let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a **partition** of n, denoted by $\lambda \vdash n$, with parts listed in nonincreasing order.

First, we give terminology needed to define the basis of Schur functions, a particularly important symmetric function basis when studying representations of the symmetric group. A **Young diagram** of shape $\lambda \vdash n$ is a collection of n boxes arranged into rows with λ_i boxes in row i and with the rows left-justified. A **skew-shape** λ/μ is obtained from the Young diagram of shape λ by deleting the boxes in a shape μ, assuming the shape μ is entirely contained in the shape λ. A **semi-standard Young tableau** of shape λ is a Young diagram of shape λ with the boxes filled with positive integers that increase weakly in rows and strictly in columns. A semi-standard Young tableau of skew-shape is defined similarly. Figure A-1 gives an example of a semi-standard Young tableau of skew-shape λ/μ for $\lambda = (4, 4, 3)$ and $\mu = (3, 1)$. If we let $m_i(T)$ be the number of occurences of i in a semi-standard Young

			3
	1	5	5
2	2	6	

Figure A-1: A semi-standard Young tableau of skew-shape $(4,4,3)/(3,1)$

tableau T, then the partition comprised of the $m_i(T)$ listed in nonincreasing order is called the **content** of T.

1. The **monomial symmetric function** m_λ is given by $m_\lambda = \sum x_{i_1}^{\lambda_1} \dots x_{i_k}^{\lambda_k}$, summing over all possible ordered sets of k distinct positive integers $i_1, \dots, i_k$.

2. The **elementary symmetric function** e_λ is given by $e_n = \sum_{a_1 < \dots < a_n} x_{a_1} \dots x_{a_n}$ and $e_\lambda = e_{\lambda_1} \dots e_{\lambda_k}$ for $\lambda = (\lambda_1, \dots, \lambda_k)$. For example, $e_3 = x_1x_2x_3 + x_1x_2x_4 + \dots + x_2x_3x_4 + \dots$ and $e_{(3,2)} = e_3e_2 = (x_1x_2x_3 + \dots)(x_1x_2 + \dots)$.

3. The **complete homogeneous symmetric function** h_λ is given by $h_n = \sum_{a_1 \leq \dots \leq a_n} x_{a_1} \dots x_{a_n}$ and $h_\lambda = h_{\lambda_1} \dots h_{\lambda_k}$ for $\lambda = (\lambda_1, \dots, \lambda_k)$.

4. The **Schur symmetric function** s_λ is given by $s_\lambda = \sum_{SSYT of shape \lambda} \prod_{i>0} x_i^{(m(i)}$ where $m(i)$ is the multiplicity of the number i in a semi-standard Young tableaux of shape λ. If we restrict to $x_1, \dots, x_n$, then another formulation which makes it more clear that this is a symmetric function is

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}$$

where $a_{\lambda+\delta}$ is the $n \times n$ determinant $|x_i^{n-j+\lambda_j}|$ with entry $x_i^{n-j+\lambda_j}$ in position (i,j).

The **skew-Schur function** $s_{\lambda/\mu}$ is defined similarly, but summing over semi-standard Young tableaux of skew-shape λ/μ. For example, the semi-standard Young tableau in Figure A-1 contributes $x_1x_2^2x_3x_5^2x_6$ to $s_{(4,4,2)/(3,1)}$. Every Schur function is also a skew-Schur function with $\mu = (0)$, but the skew-Schur functions are not all linearly independent. Note that h_n equals the Schur function of shape a single row while e_n is the Schur function of shape a single column.

Proposition A.4.1 *There is a symmetric function involution ω such that $\omega e_\lambda = h_\lambda$ and more generally $\omega s_\lambda = s_{\lambda'}$ where $s_{\lambda'}$ is the Schur function of conjugate (i.e. transpose) shape to λ. This involution is a homomorphism.*

Theorem A.4.1 *The Jacobi-Trudi identity asserts that s_λ is the $k \times k$ determinant $|h_{\lambda_i - i + j}|$ where λ has k parts and $h_{\lambda_i - i + j}$ is the entry in position (i, j).*

Theorem A.4.2 *The Littlewood-Richardson Rule gives us a formula for multiplying Schur functions,*

$$s_\mu s_\nu = \sum_{\lambda \vdash n} c^\lambda_{\mu,\nu} s_\lambda$$

where $c^\lambda_{\mu,\nu}$ is a nonnegative integer called a Littleword-Richardson coefficient and $|\mu| + |\nu| = n$. (cf. [Sa, p.173]) The skew-Schur functions are Schur-positive, because they satisfy

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

Let the **cycle index** of λ, denoted by z_λ, be $\prod_{i=1}^n (m_i)!(i^{m_i})$ where m_i counts the number of occurences of i in λ. The Schur functions are related to the power-sum symmetric functions by the identity

$$s_\lambda = \sum_\mu \chi^\lambda(\mu) \frac{p_\mu}{z_\mu}$$

with $\chi^\lambda(\mu)$ equalling the character of any permutation of cycle-type μ in the irreducible symmetric group representation indexed by the partition λ.

In particular, if the character χ of a symmetric group representation is written as a sum of irreducible characters $\chi = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda$, then the **Frobenius characteristic** of χ is the symmetric function $\sum_{\lambda \vdash n} c_\lambda s_\lambda$.

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